285.

ON THE SYSTEM OF CONICS HAVING DOUBLE CONTACT WITH EACH OTHER.

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CONSIDER the conics which pass through four points coinciding two and two together; the two points of each pair of coincident points are to be regarded as lying in a line which will be a tangent to the conic, and the system is thus a system of conics touching the same two lines at the same two points: or if we replace the two lines by a conic of the system, it is the system of conics having double contact with a given conic at the same two points. The lines may be spoken of as the tangents, and the points as the ineunts of the system; the line joining the two points is the axis, and the point of intersection of the two lines, the pole. It is assumed that the system contains real conics; the pole and axis must consequently be real, but the tangents and ineunts may be either all of them real or else all of them imaginary—we may for shortness say that in the former case the double contact is real, and that in the latter case it is imaginary. Consider first the case of a real double contact, the system of conics commences with an ellipse differing only slightly from the finite portion of the axis included between the ineunts, this ellipse being the first of a series of ellipses the last of which only slightly differs from a parabola, we have then a parabola, then a hyperbola differing only slightly from the parabola, and which is the first of a series of hyperbolas of which the last differs only slightly from the two tangents: all the before-mentioned conics are included within the angles at the pole or vertex of the triangle, viz. the ellipses and parabola within the angle towards the axis or base, and the hyperbolas, one branch of each of them within this angle and the other branch within the opposite angle. And it will be convenient to term these conics the "Multiform Series." Passing over the intermediate case, we come to a hyperbola differing only slightly from the two tangents, the first of a series of hyperbolas of which the last differs only slightly from the infinite portions, not included between the ineunts, of the axis; all these hyperbolas lying without the angles at the pole or vertex of the triangle. The last-mentioned hyperbolas may be termed the "Uniform Series." Passing over another intermediate case we return to the multiform series, the entire system of conics forming in fact a complete cycle. The intermediate case forming the transition from the multiform to the uniform series is a conic which considered as generated by a point is the pair of tangents, but which considered as enveloped by a line is a pair of points coincident with the pole. The intermediate case forming the transition from the uniform series to the multiform series is a conic which considered as generated by a point is a pair of lines coincident with the axis, but considered as enveloped by a line is the pair of ineunts.

When the double contact is imaginary, the conics of the system may still be considered as forming a multiform and a uniform series, but the uniform series consists entirely of imaginary conics. The multiform series commences with an ellipse differing only slightly from the pole; this is the first of a series of ellipses of which the last differs only slightly from a parabola, we have next the parabola, and then a series of hyperbolas, the first of which differs only slightly from the parabola, and the last of which differs only slightly from the axis. The ellipses and parabolas lie all of them on the same side of the axis; the hyperbolas lie, one branch of each on the one side of the axis, and the other branch on the other side. The uniform series (being imaginary) of course does not admit of description. The intermediate or transition cases are the same as for a real double contact, with only the variation occasioned by the tangents and ineunts being imaginary. The conics of the entire system are considered (as in the case of a real double contact) to form a cycle, but here part of the cycle, viz. the uniform series, is imaginary.

Suppose that M and U stand for the multiform and uniform series, MU for the transition form from the multiform to the uniform series, and UM for the transition form from the uniform to the multiform series. The cycle may be represented as in the figure.

$$\begin{tikzcd} & MU \arrow[dr] & \\ M \arrow[ur] & & U \arrow[dl] \\ & UM \arrow[ul] & \end{tikzcd}$$

Call either of the conics MU, UM the centre of the cycle, and say that the cycle is arranged line-wise when MU is considered as the centre, and point-wise when UM is considered as the centre. We may pass from one conic of the cycle to another incentrically or excentrically, i.e. by passing towards the centre or away from the centre. The intermediates of two conics are the conics passed through in going from the first to the second incentrically, the extramediates are those passed through in going from the first to the second excentrically. In speaking of the extramediates or the intermediates of a single conic, such conic is considered as the first conic and the second conic is understood to be the centre. The nearer extramediates are those passed through previous to reaching an extremity of the cycle, the further extramediates are those passed through subsequent to reaching an extremity of the cycle. It is clear that the intermediates and the nearer extramediates make up the series multiform or uniform to which the conic belongs, and that the further extramediates are the other series; and moreover that if the cycle be arranged line-wise and point-wise successively, the intermediates of the one arrangement are the nearer extramediates of the other arrangement and *vice versâ;* the further extramediates of each arrangement being the same.

Two points of a conic may be said to be conjunctive with respect to a given line when it is possible to pass from the one to the other along the curve without crossing the line and disjunctive in the contrary case—it being understood that both the parabola and the hyperbola are to be treated as closed curves, viz. the points at infinity of the parabola are to be considered as one and the same point, and so the points at each extremity of either asymptote of the hyperbola are to be considered as one and the same point—and in like manner two tangents of a conic are said to be conjunctive with respect to a given point, when it is possible by the revolution of the one tangent to arrive at the other tangent without sweeping through the point in question, and disjunctive in the contrary case.

Consider now any conic of the series, and call this simply the *conic.* Take upon the conic any two points a, a' and joining these with a variable point in the axis, let the lines so obtained meet the conic in the points b', b (so that ab' and $a'b$ meet on the axis). We have thus on the conic a series of points $a, b, c, \ldots$ and a second series $a', b', c', \ldots$ which are homographically related to each other, and which possess the property that taking any two points b, c of the first series and the corresponding points b', c' of the second series, the lines bc', $b'c$ meet on the axis. It is immaterial how the point a is chosen, but a being chosen at pleasure, the system will obviously depend on the way in which the point a' is chosen; so that the points of a conic may be considered as homographically related to each other in an infinity of different ways. The reciprocal construction of course applies to the tangents of a conic, so that the tangents of a conic may be considered as homographically related to each other in an infinity of different ways: and not only this, but it is clear that the tangents at points homographically related to each other are also homographically related to each other, and *vice versâ.*

The line joining corresponding points envelopes a conic, one of the conics of the system, and which may for shortness be spoken of simply as the *envelope.* The point of intersection of corresponding tangents generates a conic, one of the conics of a system, and which may in like manner be spoken of simply as the *locus.*

We may now enunciate the following theorem: Let the system be arranged line-wise, the envelope is an extramediate of the conic; viz. if a pair of corresponding points of the conic (all pairs have in this respect the same property) be conjunctive with respect to the axis, a nearer extramediate, but if disjunctive, then a further extramediate.

And again:

Let the system be arranged point-wise, the locus is an extramediate of the conic; viz. if a pair of corresponding tangents of the conic (all pairs have in this respect the same property) be conjunctive with respect to the pole, a nearer extramediate, but if disjunctive, then a further extramediate.

It is however proper to remark that when the double contact is imaginary, the pair of corresponding points or tangents (in fact any pair of points or tangents) is necessarily conjunctive, so that in this case the second alternatives in the two theorems have no application, and the envelope or locus is always a nearer extramediate. When the double contact is real, the pair of corresponding points or tangents may be either conjunctive or disjunctive, and the character (nearer or further) of the extramediate is determined accordingly.

The double contact being either real or imaginary, we may as a limiting case assume that the corresponding points or tangents coincide, the envelope or locus is in this case the conic itself. Suppose that the double contact is real, we have here two other limiting cases, viz. first, one of a pair (and therefore of each pair) of corresponding points or tangents may be situate in the axis or pass through the pole; this is the limit between the two cases of the pair of corresponding points or tangents being conjunctive and disjunctive, and therefore the envelope or locus must be the conic which is the limit between the nearer extramediates and the further extramediates, i.e. the envelope is the pair of ineunts and the locus is the pair of tangents.

Secondly, the line through a pair of corresponding points may pass through the pole, or the point of intersection of a pair of corresponding tangents may lie in the axis; the envelope or locus is here the furthest of the extramediates, viz. the envelope is the pair of points coincident with the pole, the locus is the pair of lines coincident with the axis. If considering one of the pair of corresponding points or lines as fixed, the other point or line passes through the last-mentioned limiting position, the envelope or locus returns back through the series of further extramediates.

In the statement of the preceding theorems, the envelope and locus have been considered separately, but we may if we please consider the locus as generated by the point of intersection of the tangents at the corresponding points: the locus and envelope are in this case reciprocal polars with respect to the conic; it should be noticed that the pair of corresponding points and the pair of corresponding tangents are either both conjunctive, or else both disjunctive.

2, *Stone Buildings, W.C.*, 3*rd June*, 1859.