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ON A NEW ANALYTICAL REPRESENTATION OF CURVES IN SPACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. III. (1860), pp. 225—236.]

THE ordinary analytical representation of a curve in space by the equations of two surfaces passing through the curve is, even in the case where the curve is the complete intersection of the two surfaces, inappropriate as involving the consideration of surfaces which are extraneous to the curve; and the objection becomes more serious when the curve is not the complete intersection of any two surfaces; for in this case the curve can only be represented in conjunction with a curve or curves extraneous to itself. A curve in space can be in a mode represented by means of the equation of the developable surface having the curve for its edge of regression; but this corresponds to the representation of a plane curve by means of its equation in line coordinates; a representation which is very useful in addition to, but which is not to be substituted for, the equation in point coordinates. It occurred to me some years ago that it might be advantageous to represent a curve in space by means of the cone passing through the curve and having for its vertex an arbitrary point([1]); and although I have not advanced beyond the first steps of the theory, the results which I have obtained may, I think, be interesting to geometers. The conclusion is that a curve in space may be represented by a homogeneous equation $V=0$ between six coordinates (p, q, r, s, t, u) such that $ps+qt+ru=0$; the function V is moreover such that in virtue of this relation between the coordinates, and of the equation $V=0$ itself, we have

$$d_pV\,.\,d_sV+d_qV\,.\,d_tV+d_rV\,.\,d_uV=0,$$

[1] See my paper "On the cones which pass through a curve of the third order in space," *Phil. Mag.*, t. XII. (1856), p. 20, [200], where a curve of the third order in space is in effect so represented.

or what is the same thing we have identically

$$d_p V . d_s V + d_q V . d_t V + d_r V . d_u V = LV + M(ps + qt + ru),$$

where L and M are functions of p, q, r, s, t, u. But the converse proposition, viz. any equation whatever $V=0$, where V satisfies the condition just referred to represents a curve in space, is not true; it would obviously be an important point in the theory to ascertain what further conditions must be satisfied by the function V.

The establishment of the foregoing results is very easy; in fact if x, y, z, w are current coordinates of the ordinary kind (point-coordinates) and $\alpha, \beta, \gamma, \delta$ the coordinates of an arbitrary point; the equation of any cone whatever having for its vertex the point $(\alpha, \beta, \gamma, \delta)$ may be represented by a homogeneous equation between the six determinants of the matrix

$$\begin{vmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{vmatrix}$$

or if we write

$$p = \gamma y - \beta z, \quad s = \delta x - \alpha w,$$
$$q = \alpha z - \gamma x, \quad t = \delta y - \beta w,$$
$$r = \beta x - \alpha y, \quad u = \delta z - \gamma w,$$

values which it is well known give identically

$$ps + qt + ru = 0,$$

then the cone will be represented by a homogeneous equation

$$V = 0$$

between the six coordinates (p, q, r, s, t, u). It remains to find the conditions in order that all the cones so represented, viz. the cones obtained by giving any values whatever to the arbitrary quantities $\alpha, \beta, \gamma, \delta$ which enter implicitly into the coordinates p, q, r, s, t, u, pass through one and the same curve; for when this is the case, the equation $V=0$ may be properly considered as the equation of the curve.

Assume then that all the cones pass through the same curve; if we give to one of the arbitrary quantities $\alpha, \beta, \gamma, \delta$, say α, the infinitesimal variation $d\alpha$, then the function V becomes $V + d_\alpha V . d\alpha$, and each of the equations $V=0$, $V + d_\alpha V . d\alpha = 0$ belongs to a cone passing through the curve; the equation $d_\alpha V = 0$ is therefore the equation of a surface passing through the curve; and in like manner the four equations

$$d_\alpha V = 0, \quad d_\beta V = 0, \quad d_\gamma V = 0, \quad d_\delta V = 0$$

are each of them the equation of a surface passing through the curve, or these equations must be simultaneously satisfied for all the points of the curve, they must consequently reduce themselves to two independent relations. But V is given as a function of p, q, r, s, t, u, through which quantities it is a function of $\alpha, \beta, \gamma, \delta$; the last-mentioned four equations become therefore

$$\begin{aligned} 0 &= \quad . \quad -d_rV\,.\,y + d_qV\,.\,z - d_sV\,.\,w, \\ 0 &= d_rV\,.\,x \quad . \quad -d_pV\,.\,z - d_tV\,.\,w, \\ 0 &= -d_qV\,.\,x + d_pV\,.\,y \quad -d_uV\,.\,w, \\ 0 &= d_sV\,.\,x + d_tV\,.\,y + d_uV\,.\,z \quad . \quad , \end{aligned}$$

and from the first, second, and third equation, or any other combination of three equations, we obtain at once the condition

$$d_pV\,.\,d_sV + d_qV\,.\,d_tV + d_rV\,.\,d_uV = 0,$$

so that if this condition is satisfied, the four equations do in fact reduce themselves to two independent equations; the condition in question is thus shown to be necessary. But the condition only implies that all the cones having their vertices in the neighbourhood of the point $(\alpha, \beta, \gamma, \delta)$ pass through one and the same curve; this will be the case for instance for a series of cones all of them circumscribed about one and the same surface, those having their vertices in the neighbourhood of the point $(\alpha, \beta, \gamma, \delta)$ will all pass through the curve of contact with the surface of the cone having for its vertex the point in question. But the curve of contact is not a fixed curve for all positions of the vertex, and the condition before referred to is consequently insufficient.

It may be noticed that the systems p, q, r and s, t, u are not similar to each other and that the six coordinates cannot be in any way divided into two systems which are similar to each other: the symmetry of the coordinates is in fact that of the vertices (or sides) of a complete quadrilateral (or quadrangle); thus we may divide the coordinates into two sets in a fourfold manner as follows:

$$\begin{aligned} &u, t, p\,;\;\; r, q, s, \\ &s, q, u\,;\;\; p, t, r, \\ &r, t, s\,;\;\; u, q, p, \\ &p, q, r\,;\;\; s, t, u, \end{aligned}$$

where each left-hand set corresponds to three vertices forming a triangle and each right-hand set to the remaining three vertices *in lineo*. It may be noticed also that if in the equation $V = 0$ of any curve in space we substitute for p, q, r, s, t, u, their values, and equate to zero the coefficients of the different powers and products of $\alpha, \beta, \gamma, \delta$, each of the equations so obtained will belong to a surface passing through the curve, and the entire system of these equations will be equivalent to two relations only between the coordinates x, y, z, w. But any two of these surfaces will not in general intersect only in the curve, i.e. the curve will not be the complete intersection of any two of the surfaces. It may be added that the equation of any other surface whatever through the curve will be obtained by equating to zero a syzygetic function of the functions which equated to zero give the surfaces first referred to.

As an example of the theory, the equation, in the new coordinates, of a line in space will be

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where the constants are such that

$$AF + BG + CH = 0.$$

This may be verified as follows: suppose that the line in question is given as the line of junction of the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$; then the cone through the line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is nothing else than the plane through the three points; its equation is therefore

$$\begin{vmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \\ \alpha'', & \beta'', & \gamma'', & \delta'' \end{vmatrix} = 0,$$

which may be written

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

the values of A, &c. being

$$A = \alpha'\delta'' - \alpha''\delta', \quad F = \beta'\gamma'' - \beta''\gamma',$$
$$B = \beta'\delta'' - \beta''\delta', \quad G = \gamma'\alpha'' - \gamma''\alpha',$$
$$C = \gamma'\delta'' - \gamma''\delta', \quad H = \alpha'\beta'' - \alpha''\beta',$$

which in fact satisfy the relation

$$AF + BG + CH = 0.$$

So again if the line is given as the intersection of the planes

$$ax + by + cz + dw = 0,$$
$$a'x + b'y + c'z + d'w = 0,$$

then the equation of the cone (plane) through this line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is

$$(ax + by + cz + dw)(a'\alpha + b'\beta + c'\gamma + d'\delta) - (a'x + b'y + c'z + d'w)(a\alpha + b\beta + c\gamma + d\delta) = 0,$$

which is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

the values of A, &c. being

$$A = bc' - b'c, \quad F = ad' - a'd,$$
$$B = ca' - c'a, \quad G = bd' - b'd,$$
$$C = ab' - a'b, \quad H = cd' - c'd,$$

which also satisfy the relation

$$AF + BG + CH = 0.$$

I annex the following further investigation: let p', q', r', s', t', u' and $p'', q'', r'', s'', t'', u''$ be what p, q, r, s, t, u become when $\alpha, \beta, \gamma, \delta$ are changed into $\alpha', \beta', \gamma', \delta'$ and $\alpha'', \beta'', \gamma'', \delta''$ respectively: the line represented by the equation

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

is also represented by the equations

$$Ap' + Bq' + Cr' + Fs' + Gt' + Hu' = 0,$$
$$Ap'' + Bq'' + Cr'' + Fs'' + Gt'' + Hu'' = 0,$$

which, reverting to the coordinates x, y, z, w, may be written

$$\mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'w = 0,$$
$$\mathfrak{A}''x + \mathfrak{B}''y + \mathfrak{C}''z + \mathfrak{D}''w = 0,$$

where

$$\begin{array}{ll|ll} \mathfrak{A}' = & \;\;\cdot\;\; CB' - B\gamma' + F\delta', & \mathfrak{A}'' = & \;\;\cdot\;\; C\beta'' - B\gamma'' + F\delta'', \\ \mathfrak{B}' = & - C\alpha' \;\;\cdot\;\; + A\gamma' + G\delta', & \mathfrak{B}'' = & - C\alpha'' \;\;\cdot\;\; + A\gamma'' + G\delta'', \\ \mathfrak{C}' = & B\alpha' - A\beta' \;\;\cdot\;\; + H\delta', & \mathfrak{C}'' = & B\alpha'' - A\beta'' \;\;\cdot\;\; + H\delta'', \\ \mathfrak{D}' = & - F\alpha' - G\beta' - H\gamma' \;\;\cdot\;\; , & \mathfrak{D}'' = & - F\alpha'' - G\beta'' - H\gamma'' \;\;\cdot\;\; , \end{array}$$

in which form the equations represent two planes each of them through the given line: the equation of the cone (plane) through the given line and the arbitrary point $(\alpha, \beta, \gamma, \delta)$ is

$$(\mathfrak{A}'x + \mathfrak{B}'y + \mathfrak{C}'z + \mathfrak{D}'w)(\mathfrak{A}''\alpha + \mathfrak{B}''\beta + \mathfrak{C}''\gamma + \mathfrak{D}''\delta)$$
$$- (\mathfrak{A}'\alpha + \mathfrak{B}'\beta + \mathfrak{C}'\gamma + \mathfrak{D}'\delta)(\mathfrak{A}''x + \mathfrak{B}''y + \mathfrak{C}''z + \mathfrak{D}''w) = 0,$$

or, developing,

$$(\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}')\,p + (\mathfrak{C}'\mathfrak{A}'' - \mathfrak{C}''\mathfrak{A}')\,q + (\mathfrak{A}'\mathfrak{B}'' - \mathfrak{A}''\mathfrak{B}')\,r$$
$$+ (\mathfrak{A}'\mathfrak{D}'' - \mathfrak{A}''\mathfrak{D}')\,s + (\mathfrak{B}'\mathfrak{D}'' - \mathfrak{B}''\mathfrak{D}')\,t + (\mathfrak{C}'\mathfrak{D}'' - \mathfrak{C}''\mathfrak{D}')\,u = 0.$$

Now

$$\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}' = (-C\alpha' + A\gamma' + G\delta')(B\alpha'' - A\beta'' + H\delta'')$$
$$- (-C\alpha'' + A\gamma'' + G\delta'')(B\alpha' - A\beta' + H\delta'),$$

which putting

$$\beta'\gamma'' - \beta''\gamma' = p, \quad \alpha'\delta'' - \alpha''\delta' = s,$$
$$\gamma'\alpha'' - \gamma''\alpha' = q, \quad \beta'\delta'' - \beta''\delta' = t,$$
$$\alpha'\beta'' - \alpha''\beta' = r, \quad \gamma'\delta'' - \gamma''\delta' = u,$$

become

$$A^2p_1 + ABq_1 + ACr_1 - (BG + CH)\,s_1 + AGt_1 + AHu_1,$$

or as it may be written

$$A\,(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1) - (AF + BG + CH)\,s_1.$$

Instead of writing at once $AF + BG + CH = 0$, I denote it by ∇ and I write also

$$Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1 = \Omega.$$

The series of coefficients $\mathfrak{B}'\mathfrak{C}'' - \mathfrak{B}''\mathfrak{C}''$, &c. is

$$A\Omega - \nabla s_1,\quad B\Omega - \nabla t_1,\quad C\Omega - \nabla u_1,$$
$$F\Omega - \nabla p_1,\quad G\Omega - \nabla q_1,\quad H\Omega - \nabla r_1,$$

and the required equation, restoring for Ω, ∇ their values, is

$$(Ap + Bq + Cr + Fs + Gt + Hu)(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + H_1) - (AF + BG + CH)(ps_1 + p_1 s + qt_1 + q_1 t + ru_1 + r_1 u) = 0,$$

which in virtue of $AF + BG + CH = 0$ becomes

$$(Ap + Bq + Cr + Fs + Gt + Hu)(Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1) = 0.$$

The equation

$$Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1 = 0$$

would imply that the two points $(\alpha', \beta', \gamma', \delta')$, $(\alpha'', \beta'', \gamma'', \delta'')$ are in the same plane with the line

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

(or, what is the same thing, that this line is intersected by the line through the two points); in this exceptional case, the planes determined by the given line and the two points respectively are one and the same plane, and they do not by their intersection determine the given line. But in every other case the factor $Ap_1 + Bq_1 + Cr_1 + Fs_1 + Gt_1 + Hu_1$ is not equal to zero, and the foregoing equation becomes

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

which is as it should be.

In what precedes it has been shown that the equation of the line through the points $(\alpha', \beta', \gamma', \delta')$ and $(\alpha'', \beta'', \gamma'', \delta'')$ is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where

$$A = \alpha'\delta'' - \alpha''\delta',\quad F = \beta'\gamma'' - \beta''\gamma',$$
$$B = \beta'\delta'' - \beta''\delta',\quad G = \gamma'\alpha'' - \gamma''\alpha',$$
$$C = \gamma'\delta'' - \gamma''\delta',\quad H = \alpha'\beta'' - \alpha''\beta',$$

and that the equation of the line of intersection of the planes $ax + by + cz + dw = 0$, $a'x + b'y + c'z + d'w = 0$ is

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$

where

$$A = bc' - b'c,\quad F = ad' - a'd,$$
$$B = ca' - c'a,\quad G = bd' - b'd,$$
$$C = ab' - a'b,\quad H = cd' - c'd.$$

It may be added that the condition of intersection of the two lines

$$Ap + Bq + Cr + Fs + Gt + Hu = 0,$$
$$A'p + B'q + C'r + F's + G't + H'u = 0,$$

is

$$AF' + A'F + BG' + B'G + CH' + C'H = 0,$$

and any other problems in relation to the line, for instance to find the condition that a line may pass through a given point or lie in a given plane, &c. may also be solved by means of the new coordinates.

A curve of the second order in space is either a pair of lines or a plane conic, each of which species may degenerate into a pair of intersecting lines. If we write

$$W = Ap + Bq + Cr + Fs + Gt + Hu,$$
$$W' = A'p + B'q + C'r + F's + G't + H'u,$$

then the equation of the pair of lines will be $V = WW' = 0$, and it is worth while to show that this value of V satisfies the fundamental equation $d_pV . d_sV + d_qV . d_tV + d_rV . d_uV = 0$; we have in fact $d_pV = AW' + A'W$, &c. and thence the left-hand side is

$$W'^2 (AF + BG + CH) + WW' (AF' + A'F + BG' + B'G + CH' + C'H) + W^2 (A'F' + B'G' + G'H'),$$

which in fact vanishes in virtue of the relations

$$AF + BG + CH = 0, \quad A'F' + B'G' + C'H' = 0, \quad WW' = 0.$$

The like proof applies to any curve made up of two or more curves.

Consider in the next place the plane conic given by the equations

$$ax + by + cz + dw = 0,$$
$$x^2 + y^2 + z^2 + w^2 = 0.$$

To find the equation of the cone having for its vertex the point $(\alpha, \beta, \gamma, \delta)$ and passing through the conic we must, according to Joachimsthal's method, substitute in the two equations for x, y, z, w the values $\lambda x + \mu\alpha$, $\lambda y + \mu\beta$, $\lambda z + \mu\gamma$, $\lambda w + \mu\delta$, and from the resulting equations eliminate λ, μ. The two equations become

$$\lambda^2 (x^2 + y^2 + z^2 + w^2) + 2\lambda\mu (\alpha x + \beta y + \gamma z + \delta w) + \mu^2 (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 0,$$
$$\lambda \, (ax + by + cz + dw) + \mu \, (\alpha x + \beta y + \gamma z + \delta w) = 0,$$

and thence eliminating λ, μ we find

$$\begin{aligned} &(a\alpha + b\beta + c\gamma + d\delta)^2 (x^2 + y^2 + z^2 + w^2) \\ &- 2 (a\alpha + b\beta + c\gamma + d\delta) (ax + by + cz + dw) (\alpha x + \beta y + \gamma z + \delta w) \\ &+ (ax + by + cz + dw)^2 (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = 0, \end{aligned}$$

where the left-hand side is as it should be a function of p, q, r, s, t, u; the equation may in fact be written

$$\left.\begin{array}{l} a^2(\ .\quad q^2+r^2+s^2) \\ +\ \ b^2(p^2\ \ .\ +r^2+t^2) \\ +\ \ c^2(p^2+q^2\ .\ +u^2) \\ +\ \ d^2(s^2+t^2+u^2\ \ .\ ) \\ +2bc\,(-qr+tu) \\ +2ca\,(-rp+us) \\ +2ab\,(-pq+st) \\ +2ad\,(\ \ qu-rt) \\ +2bd\,(\ \ rs-pu) \\ +2cd\,(\ \ pt-qs) \end{array}\right\}=0,$$

which treated as a quadric in (a, b, c, d) may be written

$$\left(\begin{array}{c|c|c|c} q^2+r^2+s^2 & -pq+st & -rp+us & qu-rt \\ \hline -pq+st & p^2+r^2+t^2 & -qr+tu & rs-pu \\ \hline -rp+us & -qr+tu & p^2+q^2+u^2 & pt-qs \\ \hline qu-rt & rs-pu & pt-qs & s^2+t^2+u^2 \end{array}\right)(a, b, c, d)^2=0,$$

or treated as a quadric in (p, q, r, s, t, u), in the form

$$\left(\begin{array}{c|c|c|c|c|c} b^2+c^2 & -ab & -ac & . & cd & -bd \\ \hline -ba & c^2+a^2 & -bc & -cd & . & ad \\ \hline -ca & -cb & a^2+b^2 & bd & -ad & . \\ \hline . & -cd & bd & a^2+d^2 & ab & ac \\ \hline cd & . & -ad & ba & b^2+d^2 & bc \\ \hline -bd & ad & . & ca & cb & c^2+d^2 \end{array}\right)(p, q, r, s, t, u)^2=0.$$

Or again, in a form which is one of a system of four forms,

$$(b^2+c^2,\ c^2+a^2,\ a^2+b^2,\ -bc,\ -ca,\ -ab)(p, q, r)^2+(a^2+d^2,\ b^2+d^2,\ c^2+d^2,\ bc, ca, ab)(s, t, u)^2$$

$$+\,2\left\{\begin{array}{c|c|c|c} & s & t & u \\ \hline p & . & cd & -bd \\ \hline q & -cd & . & ad \\ \hline r & bd & -ad & . \end{array}\right\}=0.$$

Representing the equation by $V=0$, the function V should verify the fundamental equation $d_pV\,.\,d_sV+d_qV\,.\,d_tV+d_rV\,.\,d_sV=0$, and we in fact have

$$\begin{aligned} d_pV &= (b^2+c^2)\,p-abq-acr+ \qquad\qquad\quad cdt-bdu,\\ d_sV &= \qquad\qquad\quad -cdq+bdr+(a^2+d^2)\,s+abt+acu, \end{aligned}$$

and thence forming the product $d_pV\,.\,d_sV$, and from it the two analogous products, we find

	$d_pV\,.\,d_sV=$	$d_qV\,.\,d_tV=$	$d_rV\,.\,d_uV=$
p^2	0	$-abcd$	$+abcd$
q^2	$+abcd$	0	$-abcd$
r^2	$-abcd$	$+abcd$	0
s^2	0	$-abcd$	$+abcd$
t^2	$+abcd$	0	$-abcd$
u^2	$-abcd$	$+abcd$	0
pq	$-cd\,(b^2+c^2)$	$+cd\,(c^2+a^2)$	$-cd\,(a^2-b^2)$
qr	$-ad\,(b^2-c^2)$	$-ad\,(c^2+a^2)$	$+ad\,(a^2+b^2)$
rp	$+bd\,(b^2+c^2)$	$-bd\,(c^2-a^2)$	$-bd\,(a^2+b^2)$
su	$-bd\,(a^2+d^2)$	$+bd\,(c^2+d^2)$	$-bd\,(c^2-a^2)$
ut	$-ad\,(b^2-c^2)$	$-ad\,(c^2+d^2)$	$+ad\,(b^2+d^2)$
ts	$+cd\,(a^2+d^2)$	$-cd\,(a^2-d^2)$	$-cd\,(b^2+d^2)$
ps	$+(a^2+d^2)(b^2+c^2)$	$-a^2b^2-c^2d^2$	$-a^2c^2-b^2d^2$
pt	$+ab\,(b^2+c^2)$	$-ab\,(b^2+d^2)$	$-ab\,(c^2-d^2)$
pu	$+ac\,(b^2+c^2)$	$-ac\,(b^2-d^2)$	$-ac\,(c^2+d^2)$
qs	$-ab\,(a^2+d^2)$	$+ab\,(c^2+a^2)$	$-ab\,(c^2-d^2)$
qt	$-a^2b^2-c^2d^2$	$+(b^2+d^2)(c^2+a^2)$	$-b^2c^2-a^2d^2$
qu	$-bc\,(a^2-d^2)$	$+bc\,(c^2+a^2)$	$-bc\,(c^2+d^2)$
rs	$-ac\,(a^2+d^2)$	$-ac\,(b^2-d^2)$	$+ac\,(a^2+b^2)$
rt	$-bc\,(a^2-d^2)$	$-bc\,(b^2+d^2)$	$+bc\,(a^2+b^2)$
ru	$-a^2c^2-b^2d^2$	$-b^2c^2-a^2d^2$	$+(c^2+d^2)(a^2+b^2)$

by which the equation is verified.

I conclude with some remarks relating to a different part of the subject. It is shown by M. E. de Jonquières in his "Essai sur la generation des courbes geometriques, &c." *Mem. Prés. à l'Acad. de Paris*, t. XVI. (1858), that the equation $U=0$ of any geometrical plane curve can be presented in a variety of different ways in the form

$$\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix}=0,$$

this being in fact the fundamental theorem of his very beautiful investigations. I am not aware that it has ever been considered whether the equation $U=0$ of a plane curve can in general be represented in the form

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ P'', & Q'', & R'' \end{vmatrix}=0,$$

or in the analogous forms where the left-hand side is a determinant of a higher order. As regards surfaces, the equation $U=0$ of a geometrical surface cannot in general be presented in the form

$$\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix} = 0,$$

nor in the similar forms where the left-hand side is a determinant of a higher order. We may consequently classify surfaces of a given order according to the forms of this nature by which they can be represented, or as I propose to term it, according to their "frangibility." It is obvious that this question is immediately connected with that of the representation of curves in space by means of the ordinary coordinates of analytical geometry; for instance if we have a surface $U=0$, which can be represented in the form

$$\begin{vmatrix} P, & Q, & R \\ P', & Q', & R' \\ P'', & Q'', & R'' \end{vmatrix} = 0,$$

then we can describe upon the surface curves such as

$$\left\|\begin{matrix} P', & Q', & R' \\ P'', & Q'', & R'' \end{matrix}\right\| = 0,$$

viz. the curve so represented is the curve which in conjunction with the curve ($R'=0$, $R''=0$) makes up the complete intersection of the two surfaces $P'R''-P''R'=0$, $Q'R''-Q''R'=0$: the curve in question is not in general the complete intersection of any two surfaces. If the surface $U=0$ can be represented in the form $\begin{vmatrix} P, & Q \\ P', & Q' \end{vmatrix} = 0$, then we can describe upon the surface, curves such as the curve ($P=0$, $Q=0$) which, although it is the complete intersection of two surfaces, is not the complete intersection of the given surface $U=0$ by any other surface. But if the surface $U=0$ cannot be represented in any such form, or as we may express it, if the surface is infrangible, then it would appear that the only curves which can be described upon the surface are those which are the complete intersection of the given surface by some other surface. The question is, I think, an interesting one in the theory of surfaces, but I doubt whether much will be done in this manner as regards the theory of curves in space, and it appears to me that there is more to be hoped for from the theory previously explained in the present paper.

2, *Stone Buildings, W.C., June* 2, 1859.