886.

ON THE SURFACES WITH PLANE OR SPHERICAL CURVES OF CURVATURE.

[From the *American Journal of Mathematics*, vol. XI. (1889), pp. 71—98; pp. 293—306.]

THE theory is considered in two nearly cotemporaneous papers—Bonnet, "Mémoire sur les surfaces dont les lignes de courbure sont planes ou sphériques," *Jour. de l'École Polyt.*, t. XX. (1853), pp. 117—306, and Serret, "Mémoire sur les surfaces dont toutes les lignes de courbure sont planes ou sphériques," *Liouville*, t. XVIII. (1853), pp. 113—162. I desire to reproduce in a more compact form, and with some additional developments, the chief results obtained in these elaborate memoirs.

The basis of the theory is a theorem by Lancret, 1806. In any curve described upon a surface, the angle between the osculating planes at consecutive points is equal to the difference of the angles between the osculating planes and the corresponding tangent planes of the surface.

This includes as a particular case Joachimsthal's theorem, *Crelle*, t. XXX. (1846): If a surface have a plane curve of curvature, then at any point thereof the angle between the plane of the curve and the tangent plane of the surface has a constant value.

Bonnet and Serret each deduce the like theorem for a spherical curve of curvature, viz.: If a surface have a spherical curve of curvature, then at any point thereof the angle between the tangent plane of the sphere and the tangent plane of the surface has a constant value. Bonnet (Mémoire, p. 235) says that this follows from Lancret's theorem. Serret (Mémoire, p. 128) obtains it, by the transformation by reciprocal radius vectors, from Joachimsthal's theorem.

I remark that the theorem for a spherical curve of curvature, and (as a particular case thereof) that for a plane curve of curvature, are obtained at once from the most elementary geometrical considerations, viz. if we have (in the same plane or in different planes) the two isosceles triangles NPP', OPP' on a common base PP', then the angle OPN is equal to the angle $OP'N$. For take P, P' consecutive points on a spherical curve of curvature; then at P, P' the normals of the surface meet in a point N, and the normals (or radii) of the sphere meet in the centre O, and we have angle $OPN =$ angle $OP'N$, that is, at each of these points the inclination of the normal of the surface to the normal of the sphere has the same value; and this value being thus the same for any two consecutive points, must be the same for all points of the curve of curvature. The proof applies to the plane curve of curvature; but in this case, the fundamental theorem may be taken to be, a line at right angles to the base PP' of the isosceles triangle NPP' is equally inclined to the two equal sides NP, NP'.

A surface may have one set of its curves of curvature plane or spherical. To include the two cases in a common formula, the equation may be written

$$k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0;$$

$k=1$ in the case of a sphere, $=0$ in that of a plane; and the expression a sphere may be understood to include a plane. I write in general A, B, C to denote the cosines of the inclinations of the normal of the surface at the point (x, y, z) to the axes of coordinates (consequently $A^2+B^2+C^2=1$). Hence considering a surface, and writing down the equations

$$k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0,$$

$$(kx-a)A + (ky-b)B + (kz-c)C = l,$$

where (a, b, c, u, l) are regarded as functions of a parameter t. The first of these equations is that of a variable sphere; and the second equation expresses that at a point of intersection of the surface with the sphere, the inclination of the tangent plane of the surface to the tangent plane of the sphere has a constant value l, viz. this is a value depending only on the parameter t, and therefore constant for all points of the curve of intersection of the sphere and surface: by what precedes, the curve of intersection is a curve of curvature of the surface, and the surface will thus have a set of spherical curves of curvature.

Supposing the surface defined by means of expressions of its coordinates (x, y, z) as functions of two variable parameters, we may for one of these take the parameter t which enters into the equation of the sphere; and if the other parameter be called θ, then the expressions of the coordinates are of the form $x, y, z = x(t, \theta), y(t, \theta), z(t, \theta)$ respectively; these give equations $dx, dy, dz = a\,dt + a'd\theta,\ b\,dt + b'd\theta,\ c\,dt + c'd\theta$, where of course (a, b, c, a', b', c') are in general functions of t, θ; and we have A, B, C proportional to $bc'-b'c$, $ca'-c'a$, $ab'-a'b$, viz. the values are equal to these expressions each divided by the square root of the sum of their squares. In order that the surface may have a set of spherical curves of curvature, the above three equations must be satisfied identically by means of the values of

$$a, b, c, u, l, A, B, C, x, y, z,$$

as functions of (t, θ); and it may be seen without difficulty that we are thereby led to a partial differential equation of the first order for the determination of the surface. But I do not at present further consider this question of the determination of a surface having one set of its curves of curvature (plane or) spherical.

Suppose now that there is a second set of (plane or) spherical curves of curvature. We have in like manner

$$\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z-2v = 0,$$
$$(\kappa x-\alpha)A+(\kappa y-\beta)B+(\kappa z-\gamma)C-\lambda=0,$$

where κ is $=1$ or $=0$ according as the curves are spherical or plane, and $(\alpha, \beta, \gamma, v, \lambda)$ are functions of a variable parameter θ. We take the t of the former set of equations and the θ of these equations as the two parameters in terms of which the coordinates (x, y, z) are expressed. This being so (the former equations being satisfied as before), if these equations are satisfied identically by the values of $\alpha, \beta, \gamma, v, \lambda, A, B, C, x, y, z$ as functions of (t, θ), then the surface will have its other set of curves of curvature also spherical. It will be recollected that by hypothesis a, b, c, u, l are functions of the parameter t only, and that $\alpha, \beta, \gamma, v, \lambda$ functions of the parameter θ only. The foregoing equations, together with the assumed relations

$$A^2+B^2+C^2=1,$$
$$Adx+Bdy+Cdz=0,$$

are the "six equations" for the determination of a surface having its two sets of curves of curvature each of them (plane or) spherical.

Assuming now the values of a, b, c, l, u as functions of t, and $\alpha, \beta, \gamma, \lambda, v$ as functions of θ, the question at once arises whether we can then satisfy the six equations. These equations other than $Adx+Bdy+Cdz=0$, or say the five equations, in effect determine any five of the eight quantities $A, B, C, x, y, z, t, \theta$, in terms of the remaining three, say they determine A, B, C, t, θ as functions of x, y, z: we thus have a differential equation $Adx+Bdy+Cdz=0$, wherein A, B, C are to be regarded as given functions of (x, y, z). An equation of this form is not in general integrable; and if the equation in question be not integrable, then clearly the system of equations cannot be satisfied by any value of z as a function of (x, y), or, what is the same thing, by any values of (x, y, z) as functions of (t, θ). We thus arrive at the condition that the equation may be integrable, viz. the condition is

$$\nabla, =A\left(\frac{dB}{dz}-\frac{dC}{dy}\right)+B\left(\frac{dC}{dx}-\frac{dA}{dz}\right)+C\left(\frac{dA}{dy}-\frac{dB}{dx}\right), =0.$$

If this be satisfied, then we have an integral equation $I=0$ (containing a constant of integration which is an absolute constant) and which is, in fact, the equation of the required surface. But it is proper to look at the question somewhat differently. Supposing that the condition $\nabla=0$ is satisfied, then we have the integral equation $I=0$, and this equation, together with the five equations, in effect determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two of them, or, what is the same thing, they determine a relation between any three of these quantities. We can, from the five equations and their differentials, and from the equation $Adx + Bdy + Cdz = 0$, obtain a differential equation between any three of the eight quantities: and it has just been seen that corresponding hereto we have an integral relation between the same three quantities; that is, the condition $\nabla = 0$ being satisfied, we can from the six equations obtain between any three of the quantities $A, B, C, x, y, z, t, \theta$ a linear differential equation of the foregoing form (for instance $Zdz + Tdt + \Theta d\theta = 0$, where Z, T, Θ are given functions of z, t, θ) which will *ipso facto* be integrable, furnishing between z, t, θ an integral equation which may be used instead of the before-mentioned integral equation $I = 0$. And we thus have (without any further integration) in all six equations which serve to determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two. It is often convenient to seek in this way for the expressions of (A, B, C and) x, y, z as functions of t, θ, in preference to seeking for the integral equation $I = 0$ between the coordinates x, y, z.

The condition $\nabla = 0$ is in fact the condition which expresses that at any point of the surface the two curves of curvature intersect at right angles. Serret (and after him Bonnet) in effect obtain the condition by the assumption of this geometrical relation, without showing that the geometrical relation is in fact the necessary condition for the coexistence of the six equations. They give the condition in the form $dx\delta x + dy\delta y + dz\delta z = 0$, where dx, dy, dz are the increments of (x, y, z) along one of the curves of curvature, and $\delta x, \delta y, \delta z$ the increments along the other curve of curvature. The equations give

$$(kx - a)\, dx + (ky - b)\, dy + (kz - c)\, dz = 0,$$

$$A dx + \quad B dy + \quad C dz = 0,$$

and similarly

$$(\kappa x - \alpha)\, \delta x + (\kappa y - \beta)\, \delta y + (\kappa z - \gamma)\, \delta z = 0,$$

$$A \delta x + \quad B \delta y + \quad C \delta z = 0.$$

We thence have

$$dx : dy : dz = B(kz - c) - C(ky - b) : C(kx - a) - A(kz - c) : A(ky - b) - B(kx - a),$$

and

$$\delta x : \delta y : \delta z = B(\kappa z - \gamma) - C(\kappa y - \beta) : C(\kappa x - \alpha) - A(\kappa z - \gamma) : A(\kappa y - \beta) - B(\kappa x - \alpha).$$

We have thus the required condition, in a form which is readily changed into

$$(A^2 + B^2 + C^2)\{(kx - a)(\kappa x - \alpha) + (ky - b)(\kappa y - \beta) + (kz - c)(\kappa z - \gamma)\}$$
$$- \{A(kx - a) + B(ky - b) + C(kz - c)\}\{A(\kappa x - \alpha) + B(\kappa y - \beta) + C(\kappa z - \gamma)\} = 0,$$

and writing herein $A^2 + B^2 + C^2 = 1$, this becomes

$$\tfrac{1}{2}\kappa\{k(x^2 + y^2 + z^2) - 2ax - 2by - 2cz\}$$
$$+ \tfrac{1}{2}k\{\kappa(x^2 + y^2 + z^2) - 2\alpha x - 2\beta y - 2\gamma z\}$$
$$+ (a\alpha + b\beta + c\gamma) - l\lambda = 0,$$

that is,

$$a\alpha + b\beta + c\gamma - l\lambda + \kappa u + kv = 0.$$

I proceed to show that this is the condition $\nabla = 0$ for the integrability of the differential equation $A dx + B dy + C dz = 0$. Writing as before

$$\nabla = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right),$$

we have from the six equations

$$AdA + BdB + CdC = 0,$$
$$(kx - a)\, dA + (ky - b)\, dB + (kz - c)\, dC = -k\,(A dx + B dy + C dz) + (Aa_1 + Bb_1 + Cc_1 + l_1)\, dt,$$
$$(\kappa x - \alpha)\, dA + (\kappa y - \beta)\, dB + (\kappa z - \gamma)\, dC = -\kappa\,(A dx + B dy + C dz) + (A\alpha' + B\beta' + C\gamma' + \lambda')\, d\theta,$$
$$(kx - a)\, dx + (ky - b)\, dy + (kz - c)\, dz = (a_1 x + b_1 y + c_1 z + u_1)\, dt,$$
$$(\kappa x - \alpha)\, dx + (\kappa y - \beta)\, dy + (\kappa z - \gamma)\, dz = (\alpha' x + \beta' y + \gamma' z + \upsilon')\, d\theta,$$

where a_1, b_1, c_1, l_1, u_1 denote derived functions in regard to t, and α', β', γ', λ', υ' derived functions in regard to θ. Putting for shortness

$$\Omega = \begin{vmatrix} A, & B, & C \\ kx - a, & ky - b, & kz - c \\ \kappa x - \alpha, & \kappa y - \beta, & \kappa z - \gamma \end{vmatrix},$$

we readily obtain

$$\begin{aligned} \Omega dA = [(\kappa y - \beta)\, C - (\kappa z - \gamma)\, B] \Big\{ &-k\,(A dx + B dy + C dz) \\ &+ \frac{Aa_1 + Bb_1 + Cc_1 + l_1}{a_1 x + b_1 y + c_1 z + u_1}\{(kx - a)\, dx + (ky - b)\, dy + (kz - c)\, dz\}\Big\} \\ - [(ky - b)\, C - (kz - c)\, B] \Big\{ &-\kappa\,(A dx + B dy + C dz) \\ &+ \frac{A\alpha' + B\beta' + C\gamma' + \lambda'}{\alpha' x + \beta' y + \gamma' z + \upsilon'}\{(\kappa x - \alpha)\, dx + (\kappa y - \beta)\, dy + (\kappa z - \gamma)\, dz\}\Big\}; \end{aligned}$$

say this is

$$\begin{aligned} \Omega dA = [(\kappa y - \beta)\, C - (\kappa z - \gamma)\, B] \Big\{ &-k\,(A dx + B dy + C dz) \\ &+ \frac{L}{P}\{(kx - a)\, dx + (ky - b)\, dy + (kz - c)\, dz\}\Big\} \\ - [(ky - b)\, C - (kz - c)\, B] \Big\{ &-\kappa\,(A dx + B dy + C dz) \\ &+ \frac{\Lambda}{\Pi}\{(\kappa x - \alpha)\, dx + (\kappa y - \beta)\, dy + (\kappa z - \gamma)\, dz\}\Big\}, \end{aligned}$$

or, introducing further abbreviations, and writing down the analogous values of ΩdB and ΩdC, we have

$$\Omega dA = [(\kappa y - \beta)\, C - (\kappa z - \gamma)\, B]\, U - [(ky - b)\, C - (kz - c)\, B]\, \Upsilon,$$
$$\Omega dB = [(\kappa z - \gamma)\, A - (\kappa x - \alpha)\, C]\, U - [(kz - c)\, A - (kx - a)\, C]\, \Upsilon,$$
$$\Omega dC = [(\kappa x - \alpha)\, B - (\kappa y - \beta)\, A]\, U - [(kx - a)\, B - (ky - b)\, A]\, \Upsilon.$$

We hence find

$$\Omega\frac{dB}{dz} = \quad [(\kappa z-\gamma)A-(\kappa x-\alpha)C]\left\{-kC+\frac{L}{P}(kz-c)\right\}$$

$$-[(kz-c)A-(kx-a)C]\left\{-\kappa C+\frac{\Lambda}{\Pi}(\kappa z-\gamma)\right\},$$

$$-\Omega\frac{dC}{dy} = -[(\kappa x-\alpha)B-(\kappa y-\beta)A]\left\{-kB+\frac{L}{P}(ky-b)\right\}$$

$$+[(kx-a)B-(ky-b)A]\left\{-\kappa B+\frac{\Lambda}{\Pi}(\kappa y-\beta)\right\}.$$

Combining these two terms, in the resulting value of $\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)$, first, the term without L or Λ is found to be

$$= -kA\{A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)\}$$

$$-k(\kappa x-a)(A^2+B^2+C^2)$$

$$+\kappa(kx-\alpha)(A^2+B^2+C^2)$$

$$+\kappa A\{A(kx-a)+B(ky-b)+C(kz-c)\},$$

which is

$$= -kA\lambda+k(\kappa x-\alpha)-\kappa(kx-a)+\kappa Al,$$

$$= \quad A(\kappa l-k\lambda)-k\alpha+\kappa a.$$

Next, the coefficient of $\frac{L}{P}$ is

$$A(kz-c)(\kappa z-\gamma)-C(\kappa x-\alpha)(kz-c)$$

$$+A(ky-b)(\kappa y-\beta)-B(\kappa x-\alpha)(ky-b),$$

which is

$$= A[(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)]$$

$$-(\kappa x-\alpha)[A(kx-a)+B(ky-b)+C(kz-c)]$$

$$= AM+(\kappa x-\alpha)l,$$

if for shortness

$$M = (kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma);$$

and similarly, the coefficient of $\frac{\Lambda}{\Pi}$ is

$$-A(kz-c)(\kappa z-\gamma)+C(kx-a)(\kappa x-\alpha)$$

$$-A(ky-b)(\kappa y-\beta)+B(kx-b)(\kappa y-\beta),$$

which is

$$= -A[(kx-a)(\kappa x-\alpha)+(ky-b)(\kappa y-\beta)+(kz-c)(\kappa z-\gamma)]$$

$$-(kx-a)[A(\kappa x-\alpha)+B(\kappa y-\beta)+C(\kappa z-\gamma)]$$

$$= -AM-(kx-a)\lambda.$$

We thus obtain

$$\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)=A(\kappa l-k\lambda)-k\alpha+\kappa a+\frac{L}{P}\{AM+(\kappa x-\alpha)l\}-\frac{\Lambda}{\Pi}\{AM+(kx-a)\lambda\},$$

and similarly

$$\Omega\left(\frac{dC}{dx}-\frac{dA}{dz}\right)=B(\kappa l-k\lambda)-k\beta+\kappa b+\frac{L}{P}\{BM+(\kappa y-\beta)l\}-\frac{\Lambda}{\Pi}\{BM+(ky-b)\lambda\},$$

$$\Omega\left(\frac{dA}{dy}-\frac{dB}{dx}\right)=C(\kappa l-k\lambda)-k\gamma+\kappa c+\frac{L}{P}\{CM+(\kappa z-\gamma)l\}-\frac{\Lambda}{\Pi}\{CM+(kz-c)\lambda\};$$

hence multiplying by A, B, C and adding, we obtain

$$\Omega\nabla=\kappa l-k\lambda-k(A\alpha+B\beta+C\gamma)+\kappa(Aa+Bb+Cc)+\frac{L}{P}(M-l\lambda)-\frac{\Lambda}{\Pi}(M-l\lambda),$$

where the first four terms are together

$$=\kappa l-k\lambda+k\{\kappa(Ax+By+Cz)-\lambda\}-\kappa\{k(Ax+By+Cz)-l\},$$

viz. these destroy each other, and the equation becomes

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(M-l\lambda).$$

But we have

$$\begin{aligned}M-l\lambda= &\ \tfrac{1}{2}\kappa\{k(x^2+y^2+z^2)-2ax-2by-2cz\}\\ &+\tfrac{1}{2}k\{\kappa(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z\}+(a\alpha+b\beta+c\gamma)-l\lambda,\end{aligned}$$

which is

$$=a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv,$$

or we find

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv),$$

viz. the condition $\nabla=0$ is

$$a\alpha+b\beta+c\gamma-l\lambda+\kappa u+kv=0,$$

the result which was to be proved.

If we consider separately the cases where the two sets of curves of curvature are each plane, the first plane and the second spherical, and each of them spherical; or say the cases PP, PS and SS, then in these cases respectively the condition is

$$a\alpha+b\beta+c\gamma-l\lambda=0,$$
$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$
$$a\alpha+b\beta+c\gamma-l\lambda+u+v=0:$$

we have, in each case, to take the italic letters functions of t and the greek letters functions of ϑ, satisfying identically the appropriate equation, but otherwise arbitrary; and then, in each case, the six equations lead to a differential equation $Adx+Bdy+Cdz=0$ (or say $Zdz+Tdt+\Theta d\theta=0$) between three variables, which equation is *ipso facto* integrable; and we thus obtain a new integral equation which, with the original five integral equations, gives the solution of the problem. The condition is, in each case, of the form $\Sigma a\alpha = 0$, the number of terms $a\alpha$ being 4, 5 or 6. Considering for instance the form

$$a\alpha + b\beta + c\gamma + d\delta + e\epsilon + f\phi = 0$$

with 6 terms, it is easy to see how such an equation is to be satisfied by values of a, b, c, d, e, f which are functions of t, and values of $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ which are functions of θ. Suppose that $t_1, t_2, \ldots$ are particular values of t, and $a_1, b_1, \ldots, f_1$; $a_2, b_2, \ldots, f_2$, &c., the corresponding values of $a, b, \ldots, f$, these values being of course absolute constants; we have $\alpha, \beta, \ldots, \phi$, functions of θ, satisfying all the equations

$$(a_1, b_1, c_1, d_1, e_1, f_1 \between \alpha, \beta, \gamma, \delta, \epsilon, \phi) = 0,$$
$$(a_2, b_2, c_2, d_2, e_2, f_2 \between \quad \text{,,} \quad) = 0,$$
&c.,

and if 6 or more of these equations were independent, the equations could, it is clear, be satisfied only by the values $\alpha = \beta = \gamma = \delta = \epsilon = \phi = 0$. To obtain a proper solution, only some number less than 6 of these equations can be independent. Suppose, for instance, that only two of the equations are independent; we then have $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ functions of θ satisfying these equations, but otherwise arbitrary; or, what is the same thing, we may take $\alpha, \beta, \gamma, \delta, \epsilon, \phi$ linear functions of $6-2, =4$ arbitrary functions, say P, Q, R, S of θ; say we have

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 \between P, Q, R, S),$$
$$\beta = (\beta_0, \ldots\ldots\ldots \between \quad \text{,,} \quad),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\phi = (\phi_0, \ldots\ldots\ldots \between \quad \text{,,} \quad),$$

where the suffixed greek letters denote absolute constants; and this being so, in order to satisfy the proposed equation $a\alpha + b\beta + c\gamma + d\delta + e\epsilon + f\phi = 0$, we must have

$$(\alpha_0, \beta_0, \gamma_0, \delta_0, \epsilon_0, \phi_0 \between a, b, c, d, e, f) = 0,$$
$$(\alpha_1, \ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,$$
$$(\alpha_2, \ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,$$
$$(\alpha_3, \ldots\ldots\ldots\ldots\ldots \between \quad \text{,,} \quad) = 0,$$

viz. a, b, c, d, e, f will then be functions of t satisfying these four equations, but otherwise arbitrary. The above is a solution for the partition $2+4$ of the number 6. We have in like manner a solution for any other partition of 6; or if we disregard the extreme cases $a = b = c = d = e = f = 0$ and $\alpha = \beta = \gamma = \delta = \epsilon = \phi = 0$, then we have in this manner solutions for the several partitions 15, 24, 33, 42 and 51 of the number 6.

But applying this theory to the actual problem, there is a good deal of difficulty as regards the enumeration of the really distinct cases. I use the letters P, S to denote that a set of curves of curvature is plane or spherical as the case may be, the surfaces to be considered are thus *PP*, *PS*, and *SS*. First, for the *PP* problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda = 0$: the two systems (a, b, c, l) and $(\alpha, \beta, \gamma, \lambda)$ are symmetrically related to each other, and instead of the solutions 13, 22 and 31, it is sufficient to consider the solutions 13 and 22. But here (a, b, c, l) are not a system of four symmetrically related functions, (a, b, c) are a symmetrical system, and l is a distinct term: and the like for the system $(\alpha, \beta, \gamma, \lambda)$. In the *PS* problem, where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u = 0$, and thus the systems (a, b, c, l, u), $(\alpha, \beta, \gamma, \lambda, 1)$ are of different forms, we should consider the solutions 14, 23, 32 and 41: but here again, in each of the systems separately, the terms are not symmetrically related to each other. Lastly, in the *SS* problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u + v = 0$: the systems $(a, b, c, l, u, 1)$ and $(\alpha, \beta, \gamma, \lambda, 1, v)$ are of the same form, it is enough to consider the solutions 15, 24 and 33; but in this case also, in each of the systems separately, the terms are not symmetrically related to each other. I do not at present further consider the question, but simply adopt Serret's enumeration.

It is to be remarked that for a developable (but not for a skew surface) the generating lines may be curves of curvature, and regarding the generating lines as plane curves we might have developables *PP* or *PS*; but a straight line is not a curve in a determinate plane, and it is better to consider the case apart from the general theory. Again, the curves of curvature of one set or those of each set may be circles; and a circle may be regarded either as a plane or a spherical curve; regarding it, however, as a spherical curve, it is a curve not in a determinate sphere. The cases in question, of the curves of curvature of the one set or of those of each set being circles, are therefore also to be considered apart from the general theory. The surfaces referred to present themselves for consideration among Serret's cases *PP*, 1^0, 2^0, 3^0; *PS*, 1^0, 2^0, 3^0, 4^0, 5^0, 6^0, 7^0; and *SS*, 1^0, 2^0, 3^0, 4^0; but they are excluded from his enumeration, and he in fact reckons in his "Conclusion," pp. 161, 162, two kinds of surfaces *PP*, three kinds *PS*, and two kinds *SS*.

It is very easily seen that, if a surface has a plane or a spherical curve of curvature, then on any parallel surface the corresponding curve is a plane or a spherical curve of curvature: and thus if a surface be *PP*, *PS*, or *SS*, then the parallel surfaces are respectively *PP*, *PS*, or *SS*. The solutions obtained include for the most part all the parallel surfaces, and thus there is no occasion to make use of this theorem; but see in the continuation of the present paper the case considered under the subheading *post*, *PS*, 4^0 = Serret's third case of *PS*.

If a surface have a plane or a spherical curve of curvature, then transforming the surface by reciprocal radius vectors (or inverting in regard to an arbitrary point), then in the transformed surface the corresponding curve is a spherical curve of curvature. Hence if a surface be *PP*, *PS* or *SS*, the transformed surface is *SS*. Conversely, as shown by Bonnet and Serret, and as will appear, every surface *SS* is in fact an inversion of a surface *PP* or *PS*.

I proceed to the enumeration, developing the theory only in regard to the two, three, and two, cases *PP*, *PS* and *SS* respectively.

PP, The Two Sets of Curves of Curvature each Plane.

The six equations are

$$\begin{aligned} A^2+B^2 \quad +C^2 \quad &=1, \\ ax+by \quad +cz \quad +u &=0, \\ Aa+Bb \quad +Cc \quad +l &=0, \\ \alpha x+\beta y \quad +\gamma z \quad +v &=0, \\ A\alpha+B\beta \quad +C\gamma \quad +\lambda &=0, \\ Adx+Bdy+Cdz \quad &=0; \end{aligned}$$

the condition is

$$a\alpha+b\beta+c\gamma-l\lambda=0,$$

not containing u or v, so that these remain arbitrary functions of t, θ respectively. The cases are

	a	b	c	l	α	β	γ	λ
PP, 1°	1	0	c	0	0	1	0	λ
PP, 2°	0	1	0	$-m$	α	$-m\lambda$	γ	λ
PP, 3°	1	0	c	mc	0	1	$m\lambda$	λ;

m is an arbitrary constant; and in the body of the table, c is an arbitrary function of t, and α, γ, λ arbitrary functions of θ.

PP, 1° is Serret's first case of PP, included in his second case.

PP, 2° gives a developable.

PP, 3° is Serret's second case of PP.

I consider the case

PP, 3° = Serret's Second Case of PP.

Writing for greater symmetry $m=g$, $\frac{1}{m}=f$, so that $fg=1$; also $m\lambda=\gamma$, and consequently $\lambda=f\gamma$, we take c and γ for the two parameters respectively, or write $c=t$, $\gamma=\theta$; also changing the letters u, v, we write

a	b	c	l	u	α	β	γ	λ	v
$=1$,	0,	t	gt	P	0	1	θ	$f\theta$	Π,

and the six equations thus are

$$\begin{aligned} A^2+B^2 \quad +C^2 &=1, \\ x+tz \quad -P &=0, \\ A+tC \quad -gt &=0, \\ y+\theta z \quad -\Pi &=0, \\ B+\theta C \quad -f\theta &=0, \\ Adx+Bdy+Cdz &=0. \end{aligned}$$

We seek for the differential equation in z, t, θ. We have

$$A^2+B^2+C^2=1,\quad A=t(g-C),\quad B=\theta(f-C),$$

and thence

$$t^2(g-C)^2+\theta^2(f-C)^2+C^2=1,$$

that is,

$$C^2(1+t^2+\theta^2)+2C(gt^2+f\theta^2)=1-g^2t^2-f^2\theta^2,$$

or multiplying by $1+t^2+\theta^2$ and completing the square,

$$\begin{aligned}\{(1+t^2+\theta^2)C-gt^2-f\theta^2\}^2 &= (1-g^2t^2-f^2\theta^2)(1+t^2+\theta^2)-(gt^2+f\theta^2)^2\\ &= \{f+(f-g)t^2\}\{g+(g-f)\theta^2\}\\ &= \frac{1}{T^2\Theta^2},\end{aligned}$$

if

$$\frac{1}{T^2}=f+(f-g)t^2,$$

$$\frac{1}{\Theta^2}=g+(g-f)\theta^2;$$

and thence, giving a determinate sign to the square root, say

$$(1+t^2+\theta^2)C=gt^2+f\theta^2-\frac{1}{T\Theta},$$

an equation which may also be written

$$C=\frac{fT-g\Theta}{f-g}.$$

In fact, observing that $\dfrac{1}{T^2}-\dfrac{1}{\Theta^2}=(f-g)(1+t^2+\theta^2)$, we deduce from the original form

$$\begin{aligned}\left(\frac{1}{T^2}-\frac{1}{\Theta^2}\right)C &= (f-g)(gt^2+f\theta^2)-\frac{f-g}{T\Theta},\\ &= g\left(\frac{1}{T^2}-f\right)-f\left(\frac{1}{\Theta^2}-g\right)-\frac{f-g}{T\Theta}\\ &= \left(\frac{g}{T}-\frac{f}{\Theta}\right)\left(\frac{1}{T}+\frac{1}{\Theta}\right),\end{aligned}$$

or throwing out the factor $\dfrac{1}{T}+\dfrac{1}{\Theta}$ and reducing, we have the required value; and thence forming the values of A and B, we have

$$A=-tT\frac{f-g}{T-\Theta},\quad B=-\theta\Theta\frac{f-g}{T-\Theta},\quad C=\frac{fT-g\Theta}{T-\Theta};$$

we have, moreover,

$$x+tz=P,\quad y+\theta z=\Pi,$$

or differentiating, and writing P_1 and Π' for the derived functions in regard to t and θ respectively,

$$dx = -tdz - zdt + P_1 dt, \quad dy = -\theta dz - zd\theta - \Pi' d\theta.$$

The equation $Adx + Bdy + Cdz = 0$ thus becomes

$$-Ttdx - \Theta\theta dy + \frac{fT - g\Theta}{f - g} dz = 0,$$

viz. this is

$$[-tT(-tdz - zdt + P_1 dt) - \theta\Theta(-\theta dz - zd\theta + \Pi' dt)](f-g) + (fT - g\Theta)\, dz = 0,$$

or collecting,

$$[\{f + (f-g)t^2\}\, T - \{g + (g - ft^2)\}\, \Theta]\, dz + (tTzdt + \theta\Theta zd\theta)(f-g)$$
$$-(tTP_1 dt + \theta\Theta\Pi' d\theta)(f-g) = 0,$$

that is,

$$\left(\frac{1}{\Theta} - \frac{1}{T}\right) dz + (f-g)\, z\, (tTdt + \theta\Theta d\theta) - (f-g)(tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

which is an integrable form as it should be; viz. the equation is

$$d\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f-g)(tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

and we obtain

$$\left(\frac{1}{T} - \frac{1}{\Theta}\right) z - (f-g)\int (tTP_1 dt + \theta\Theta\Pi' d\theta) = 0,$$

the constant of integration being considered as included in the integral. But it is proper to alter the form of the second term. Take F, Φ arbitrary functions of t, θ respectively: and writing F_1, Φ' for the derived functions: assume $P = \frac{gF_1}{T^3}$, $\Pi = \frac{f\Phi'}{\Theta^3}$; we have

$$\int (tTP_1 dt + \theta\Theta\Pi' d\theta) = \int \left(gtT\left(\frac{F_1}{T^3}\right)_1 dt + f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)' d\theta\right)$$

$$= -F + \frac{gtF_1}{T^2} - \Phi + \frac{f\theta\Phi'}{\Theta^2}.$$

In fact, this will be true if only

$$\left(-F + \frac{gtF_1}{T^2}\right)_1 = gtT\left(\frac{F_1}{T^3}\right)_1, \quad \left(-\Phi + \frac{f\theta\Phi'}{\Theta^2}\right)' = f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)',$$

which are equations of like form in t, θ respectively; it will be sufficient to verify the first of them. Effecting the differentiation, the terms in F_{11} destroy each other, and there remain only terms containing the factor F_1; throwing this out, we obtain

$$-1 + \frac{g}{T^2} + \frac{gtT_1}{T^3} = 0,$$

viz. this is

$$-1+g\{f+(f-g)t^2\}-gt^2(f-g)=0,$$

which is identically true, and the equation is thus verified.

The foregoing result is

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}=0;$$

we then have

$$x+tz-\frac{gF_1}{T^3}=0,\quad y+\theta z-\frac{f\Phi'}{\Theta^3}=0,$$

and hence, repeating also the equation for z,

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)x+(f-g)\left\{-t(F+\Phi)+\frac{ft\theta\Phi'}{\Theta^2}\right\}+\left(-1+\frac{g}{\Theta T}\right)\frac{F_1}{T^2}=0,$$

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)y+(f-g)\left\{-\theta(F+\Phi)+\frac{gt\theta F_1}{T^2}\right\}+\left(1-\frac{f}{\Theta T}\right)\frac{\Phi'}{\Theta^2}=0,$$

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}=0,$$

equations which give the values of the coordinates x, y, z in terms of the parameters t, θ. It will be recollected that $fg=1$ (f or g being arbitrary), then the values of T, Θ are

$$\frac{1}{T^2}=f+(f-g)t^2,\quad \frac{1}{\Theta^2}=g+(g-f)\theta^2,$$

and that F, Φ denote arbitrary functions of t, θ respectively. I repeat also the foregoing equations

$$A,\ B,\ C=-tT\frac{f-g}{T-\Theta},\ -\theta\Theta\frac{f-g}{T-\Theta},\ \frac{fT-g\Theta}{T-\Theta}.$$

The equations may be presented under a different form; we have

$$-tTx-\theta\Theta y+\frac{fT-g\Theta}{f-g}z+F+\Phi=0,$$

$$-fT^3(x+tz)+F_1=0,$$

$$-g\Theta^3(y+\theta z)+\Phi'=0,$$

where it will be observed that the second and third equations are the derivatives of the first equation in regard to t and θ respectively. We thus have the required surface as the envelope of the plane represented by the first equation, regarding therein t, θ as variable parameters. Moreover, the second equation (which contains only the parameter t) represents the planes of the curves of curvature of the one set; and the third equation (which contains only the parameter θ) represents the planes of the curves of curvature of the other set. It is to be observed that, from the equations for l, λ, viz. $A + tC = gt$ and $B + \theta C = f\theta$, then for any plane of the first set the inclination to a tangent plane of the surface is $= \cos^{-1} \dfrac{gt}{\sqrt{1+t^2}}$, and that for any plane of the second set the inclination is $= \cos^{-1} \dfrac{f\theta}{\sqrt{1+\theta^2}}$.

It may be remarked that the last-mentioned results may be arrived at by the consideration of an equation $Ax + By + Cz + D = 0$, where the coefficients are functions of t and θ (A a function of t only, and B a function of θ only), such that the derived equations $A_1 x + C_1 z + D_1 = 0$ and $B'x + C'z + D' = 0$ depend the former of them upon t only, and the latter of them upon θ only.

A very simple case of the equation is when $f = g = 1$; here $T = \Theta = 1$, and the surface is the envelope of the plane $z - tx - \theta y + F + \Phi = 0$.

Returning to the general form

$$-tTx - \theta\Theta y + \frac{fT - g\Theta}{f-g} z + F + \Phi = 0,$$

I transform this, by introducing therein in place of t, θ two variable parameters α, β which are such that $k\alpha = -tT$, $k\beta = \theta\Theta$ $\left(k \text{ a constant which is presently put } = \dfrac{1}{\sqrt{f-g}}\right)$; we find

$$t^2 = \frac{fk^2\alpha^2}{1-(f-g)k^2\alpha^2}, \quad \theta^2 = \frac{gk^2\beta^2}{1-(g-f)k^2\beta^2},$$

and thence

$$T = \frac{1}{\sqrt{f}}\sqrt{1-(f-g)k^2\alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}}\sqrt{1-(g-f)k^2\beta^2},$$

or putting $k = \dfrac{1}{\sqrt{f-g}}$, these last values are

$$T = \frac{1}{\sqrt{f}}\sqrt{1-\alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}}\sqrt{1+\beta^2},$$

and we hence obtain

$$\frac{fT - g\Theta}{f-g} = \frac{\sqrt{f}}{f-g}\sqrt{1-\alpha^2} - \frac{\sqrt{g}}{f-g}\sqrt{1+\beta^2},$$

$$= \frac{1}{\sqrt{f-g}}\left\{\frac{\sqrt{f}}{\sqrt{f-g}}\sqrt{1-\alpha^2} - \frac{\sqrt{g}}{\sqrt{f-g}}\sqrt{1+\beta^2}\right\},$$

say this is

$$= k\{\lambda\sqrt{1-\alpha^2} - \mu\sqrt{1+\beta^2}\},$$

where $\lambda = \dfrac{\sqrt{f}}{\sqrt{f-g}}$, $\mu = \dfrac{\sqrt{g}}{\sqrt{f-g}}$, and therefore $\lambda^2 - \mu^2 = 1$ or $\mu = \sqrt{\lambda^2 - 1}$.

Hence writing $F + \Phi = k(A + B)$, k times the sum of two arbitrary functions of α and β respectively, the equation becomes

$$\alpha x - \beta y + z\{\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2-1}\sqrt{1+\beta^2}\} + A + B = 0,$$

viz. the surface is given as the envelope of this plane considering α, β as two variable parameters. This is the solution given by Darboux, *Leçons sur la théorie générale des surfaces, &c.*, t. I., Paris, 1887, pp. 128—131. He obtains it in a very elegant manner, starting from the following theorem: Take A, A_1, &c., functions of the parameter α, and B, B_1, &c., functions of the parameter β; then, if we have identically

$$(A_1-B_1)^2+(A_2-B_2)^2+(A_3-B_3)^2=(A_4-B_4)^2,$$

the required surface will be obtained as the envelope of the plane

$$(A_1-B_1)\,x+(A_2-B_2)\,y+(A_3-B_3)\,z=A-B,$$

where A, B are two new functions of α, β respectively.

The foregoing identity is the condition in order that each sphere of the one series $(x-A_1)^2+(y-A_2)^2+(z-A_3)^2=A_4^2$ may touch each sphere of the other series $(x-B_1)^2+(y-B_2)^2+(z-B_3)^2=B_4^2$; the two series of spheres thus envelope one and the same surface which will have its curves of curvature of each set circles: viz. this will be the surface of the fourth order called Dupin's Cyclide, the normals whereof pass through an ellipse and hyperbola which are focal curves one of the other, and which contain the centres of all the spheres touching the surface along its curves of curvature. The equations of the ellipse and the hyperbola may be taken to be

$$x^2+\frac{z^2}{\lambda^2}=1,\ y=0,\ \text{and}\ y^2-\frac{z^2}{\lambda^2-1}=-1,\ x=0,$$

respectively, and we thence obtain the required PP surface as the envelope of the plane

$$\alpha x-\beta y+(\lambda\sqrt{1-\alpha^2}-\sqrt{\lambda^2-1}\,\sqrt{1+\beta^2})\,z+A+B=0.$$

The Case PP, 1^0 = *Serret's First Case of PP.*

We deduce this from the second case by writing therein $m=0$, that is, $g=0$, $f=\infty$; but it is necessary to make also a transformation upon the parameter θ, viz. in place thereof we introduce the new parameter ϕ, where $\theta^2=\dfrac{g\phi^2}{f-g\phi^2}$. This gives

$$\frac{1}{\Theta^2}=g+(g-f)\,\theta^2=g\left\{1+\frac{(g-f)\,\phi^2}{f-g\phi^2}\right\}=\frac{gf(1-\phi^2)}{f-g\phi^2},\quad \theta^2=\frac{g\phi^2}{f-g\phi^2},$$

and thence

$$\theta\Theta=\frac{\theta}{\sqrt{g+(g-f)\,\theta^2}}=\frac{\phi}{\sqrt{f}\sqrt{1-\phi^2}};\quad \frac{fT-g\Theta}{f-g}\ \text{for}\ g=0\ \text{is}\ =T.$$

We have also $T=\dfrac{1}{\sqrt{f+(f-g)\,t^2}},=\dfrac{1}{\sqrt{f}\sqrt{1+t^2}}$ when $g=0$, and substituting these values, considering Φ as a function of ϕ, and for $F+\Phi$ writing as we may do $\dfrac{F+\Phi}{\sqrt{f}}$, the equation becomes

$$\frac{-t}{\sqrt{f}\sqrt{1+t^2}}\,x-\frac{\phi y}{\sqrt{f}\sqrt{1-\phi^2}}+\frac{z}{\sqrt{f}\sqrt{1+t^2}}+\frac{F+\Phi}{\sqrt{f}}=0,$$

where the divisor $\sqrt{f}$ is to be omitted. Hence finally, instead of ϕ restoring the original letter θ, and again considering Φ as a function of θ, the equation is

$$\frac{z-tx}{\sqrt{1+t^2}}-\frac{\theta y}{\sqrt{1-\theta^2}}+F+\Phi=0,$$

viz. here F, Φ are arbitrary functions of t, θ respectively, and the surface is the envelope of this plane considering t, θ as variable.

We obtain an imaginary special form of PP, 1°, by writing in this equation $k\theta$ for θ and then putting $k=\infty$; the Φ remains an arbitrary function of the new θ, and the equation is

$$\frac{z-tx}{\sqrt{1+t^2}}+iy+F+\Phi=0,$$

($i=\sqrt{-1}$ as usual). This is, in fact, the equation which is obtained from PP, 3° by simply writing therein $g=0$ without the transformation upon θ.

PS, The Sets of Curves of Curvature, the First Plane, the Second Spherical.

The six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&ax+by+cz+u &&=0,\\
&Aa+Bb+Cc+l &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2v &&=0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\gamma)-\lambda &&=0,\\
&Adx+Bdy+Cdz &&=0.
\end{aligned}$$

The condition is

$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$

not containing v, so that this remains an arbitrary function of θ. The cases are

	a	b	c	l	u	α	β	γ	λ
PS, 1°	a	b	c	0	0	0	0	0	λ
PS, 2°	a	b	c	l	ml	0	0	0	m
PS, 3°	a	b	c	$-mc$	0	0	0	γ	$\frac{1}{m}\gamma$
PS, 4°	a	b	0	l	ml	0	0	γ	m
PS, 5°	a	b	0	0	0	0	0	γ	λ
PS, 6°	0	b	0	l	ml	α	0	γ	m
PS, 7°	a	b	0	ma	0	α	0	γ	$-\frac{1}{m}\alpha$,

where m is an arbitrary constant; in the body of the table, the other italic letters are arbitrary functions of t, and the greek letters arbitrary functions of θ.

PS, 1^0 is Serret's first case of *PS*, included in his second case.

PS, 2^0 gives developable.

PS, 3^0 is Serret's second case of *PS*.

PS, 4^0 is Serret's third case of *PS*.

PS, 5^0 gives circular sections (surfaces of revolution).

PS, 6^0 gives circular sections (tubular surfaces).

PS, 7^0 gives circular sections.

I consider

PS, 3^0 = *Serret's Second Case of PS.*

The six equations are

$$\begin{aligned} &A^2 + B^2 + C^2 = 1,\\ &ax + by + cz = 0,\\ &Aa + Bb + Cc = -cm,\\ &x^2 + y^2 + (z-m\phi)^2 = \theta + m^2\phi^2,\\ &Ax + By + C(z-m\phi) = \phi,\\ &Adx + Bdy + Cdz = 0, \end{aligned}$$

where a, b, c are assumed such that $a^2+b^2+c^2=1$. We easily obtain

$$\begin{aligned} (1-c^2)A &= -ac(C+m) - b\sqrt{\Omega},\\ (1-c^2)B &= -bc(C+m) + a\sqrt{\Omega}, \end{aligned}$$

and thence

$$aB - bA = \sqrt{\Omega},$$

where

$$\Omega = (1-c^2)(1-C^2) - c^2(C+m)^2, \ = 1 - c^2 - C^2 - 2c^2Cm - c^2m^2;$$

also

$$\begin{aligned} x\sqrt{1-c^2m^2} &= A\phi\sqrt{1-c^2m^2} + (bC-cB)\sqrt{\theta+(m^2-1)\phi^2},\\ y\sqrt{1-c^2m^2} &= B\phi\sqrt{1-c^2m^2} + (cA-aC)\sqrt{\theta+(m^2-1)\phi^2},\\ z\sqrt{1-c^2m^2} &= (C+m)\phi\sqrt{1-c^2m^2} + (aB-bA)\sqrt{\theta+(m^2-1)\phi^2}. \end{aligned}$$

We seek for the differential equation in C, t, θ. From the equation

$$Ax + By + (C - m\phi)z = \phi,$$

and attending to

$$Adx + Bdy + Cdz = 0,$$

we deduce

$$xdA + ydB + (z-m\phi)dC - (1+Cm)\phi' d\theta = 0,$$

and we have herein to substitute for dA, dB their values in terms of dC, dt, $d\theta$. We have

$$AdA + BdB = - CdC,$$
$$adA + bdB = - cdC - Q,$$

if for shortness

$$Q = Ada + Bdb + (C + m)\, dc.$$

Hence

$$\sqrt{\Omega}\, dA = (- cB + bC)\, dC - BQ,$$
$$\sqrt{\Omega}\, dB = (- aC + cA)\, dC + AQ.$$

We find without difficulty,

$$(1 - c^2)\, Q = (C + m)\, dc + (adb - bda) \sqrt{\Omega},$$

and consequently,

$$(1 - c^2) \sqrt{\Omega}\, dA = \{\ \ b\, (C + c^2 m) - ac \sqrt{\Omega}\}\, dC - B\, \{(C + m)\, dc + (adb - bda) \sqrt{\Omega}\},$$
$$(1 - c^2) \sqrt{\Omega}\, dB = \{- a\, (C + c^2 m) - bc \sqrt{\Omega}\}\, dC + A\, \{(C + m)\, dc + (adb - bda) \sqrt{\Omega}\}.$$

Substituting these values, we have

$$\begin{aligned}&\{(bx - ay)\, (C + c^2 m) - (ax + by)\, c \sqrt{\Omega}\}\, dC \\ &- (Bx - Ay)\, \{(C + m)\, dc + (adb - bda) \sqrt{\Omega}\} \\ &+ (1 - c^2) \sqrt{\Omega}\, \{(z - m\phi)\, dC - (1 + Cm)\, \phi' d\theta\} = 0,\end{aligned}$$

viz. this is

$$\begin{aligned}&\{(bx - ay)\, (C + c^2 m) - (ax + by)\, c \sqrt{\Omega} + (1 - c^2)\, (z - m\phi) \sqrt{\Omega}\}\, dC \\ &- (Bx - Ay)\, \{(C + m)\, dc + (adb - bda) \sqrt{\Omega}\} \\ &- (1 - c^2)\, (1 + Cm) \sqrt{\Omega} \phi' d\theta = 0.\end{aligned}$$

The coefficient of dC contains a term $-(ax + by + cz)\, c \sqrt{\Omega}$ which is $= 0$. Moreover, we have

$$bx - ay = - \phi \sqrt{\Omega} + \frac{C + c^2 m}{\sqrt{1 - c^2 m^2}} \sqrt{\theta + (m^2 - 1)\, \phi^2},$$

and then

$$\begin{aligned}(1 - c^2)\, (Bx - Ay) &= - c\, (C + m)\, (bx - ay) - cz \sqrt{\Omega} \\ &= - c\, (C + m) \left\{- \phi \sqrt{\Omega} + \frac{C + c^2 m}{\sqrt{1 - c^2 m^2}} \sqrt{\theta + (m^2 - 1)\, \phi^2}\right\} \\ &\quad - c \sqrt{\Omega} \left\{(C + m)\, \phi + \frac{\sqrt{\Omega} \sqrt{\theta + (m^2 - 1)\, \phi^2}}{\sqrt{1 - c^2 m^2}}\right\},\end{aligned}$$

which, observing that the terms in $C\phi \sqrt{\Omega}$ destroy each other, and that we have

$$(C + m)\, (C + c^2 m) + \Omega = (1 - c^2)\, (1 + Cm),$$

gives

$$Bx - Ay = \frac{- c\, (1 + Cm)}{\sqrt{1 - c^2 m^2}} \sqrt{\theta + (m^2 - 1)\, \phi^2},$$

and the equation becomes

$$\left\{\left(-\phi\sqrt{\Omega}+\frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right)(C+c^2m)+z\sqrt{\Omega}-(1-c^2)\,m\phi\sqrt{\Omega}\right\}dC$$

$$-c\,(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\right\}$$

$$-(1-c^2)\sqrt{\Omega}\,(1+Cm)\,\phi' d\theta=0.$$

Here the coefficient of dC is

$$=[z-(C+m)\,\phi]\sqrt{\Omega}+\frac{(C+c^2m)^2}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\,\phi^2},$$

or, substituting for $z-(C+m)\sqrt{\Omega}$ its value $=\dfrac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\,\phi^2}}{\sqrt{1-c^2m^2}}$, and observing that $\Omega+(C+c^2m)^2=(1-c^2m^2)(1-c^2)$, this coefficient is found to be

$$=\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\,\phi^2},$$

and we have

$$\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\,\phi^2}\,dC$$

$$-c\,(1+Cm)\frac{\sqrt{\theta+(m^2-1)\,\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\right\}$$

$$-(1-c^2)(1+Cm)\sqrt{\Omega}\,\phi' d\theta=0,$$

or, as this may be written,

$$\frac{1}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}\,dC}{1+Cm}-\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\}-\frac{c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}-\frac{\phi' d\theta}{\sqrt{\theta+(m^2-1)\,\phi^2}}=0,$$

where from the foregoing value of Ω we have identically

$$\Omega\,(1-m^2)=(1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2.$$

Here a, b, c are functions of t; and we have thus the required differential equation in C, t, θ.

It is convenient to multiply by the constant factor $\sqrt{1-m^2}$. The first term is an exact differential, viz. writing

$$\sin\zeta=\frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm},\text{ and therefore }\cos\zeta=\frac{\sqrt{1-m^2}\sqrt{\Omega}}{\sqrt{1-c^2}\,(1+Cm)},$$

we have

$$d\zeta=\frac{\sqrt{1-m^2}}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}\,dC}{1+Cm}+\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\},$$

as may easily be verified. And the second and third terms are obviously the differentials of a function of t and a function of θ respectively. But to obtain the integral functions, a transformation of each term is required.

First, for the term $\frac{\sqrt{1-m^2}c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$; we take a, b, c functions of t which are such that $a^2+b^2+c^2=1$; and then writing a_1, b_1, c_1 for the derived functions so that $aa_1+bb_1+cc_1=0$, we assume a′, b′, c′ = Va_1, Vb_1, Vc_1 where $\frac{1}{V^2}=a_1{}^2+b_1{}^2+c_1{}^2$; we have therefore $aa'+bb'+cc'=0$, and $a'^2+b'^2+c'^2=1$; and then writing a″, b″, c″ = bc′ − b′c, ca′ − c′a, ab′ − a′b respectively, we have

$$aa''+bb''+cc''=0,\;\; a'a''+b'b''+c'c''=0,\;\; a''^2+b''^2+c''^2=1;$$

thus a, b, c, a′, b′, c′, a″, b″, c″ are a set of rectangular coefficients. We then write

$$a,\; b,\; c=\frac{1}{\rho}(a'+mb''),\;\; \frac{1}{\rho}(b'-ma''),\;\; \frac{1}{\rho}c',$$

determining ρ so that $a^2+b^2+c^2=1$ as above, viz. we thus have

$$\rho^2=(1+cm)^2+c'^2m^2.$$

Observe that we thus have $\rho^2(1-c^2)=\rho^2-c'^2$ and $\rho^2(1-c^2m^2)=(1+cm)^2$.

Writing now

$$T=\tan^{-1}\frac{c+m}{c''\sqrt{1-m^2}},\text{ and therefore }\sin T=\frac{c+m}{\sqrt{\rho^2-c'^2}},\;\; \cos T=\frac{c''\sqrt{1-m^2}}{\sqrt{\rho^2-c'^2}},$$

we find that

$$dT=\frac{\sqrt{1-m^2}c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}.$$

The verification is somewhat long, but it is very interesting. We have

$$dT=\frac{\sqrt{1-m^2}\,\{c''dc-(c+m)\,dc''\}}{\rho^2-c'^2},$$

or observing that $c''=ab'-a'b$, $=V(ab_1-a_1b)$, $dc=c_1dt$, this is

$$dT=\frac{\sqrt{1-m^2}}{\rho^2-c'^2}\{V(ab_1-a_1b)\,c_1-(c+m)\,[V_1(ab_1-a_1b)+V(ab_{11}-a_{11}b)]\,dt\},$$

where we have

$$\frac{1}{V^2}=a_1{}^2+b_1{}^2+c_1{}^2,\text{ and therefore }-\frac{V_1}{V^3}=a_1a_{11}+b_1b_{11}+c_1c_{11};$$

also from $aa_1+bb_1+cc_1=0$, we have $a_1{}^2+b_1{}^2+c_1{}^2+aa_{11}+bb_{11}+cc_{11}=0$, and we thence obtain

$$dT=\frac{\sqrt{1-m^2}\,V^3dt}{\rho^2-c'^2}\{-(ab_1-a_1b)\,c_1(aa_{11}+bb_{11}+cc_{11})$$
$$-(c+m)\,[-(a_1a_{11}+b_1b_{11}+c_1c_{11})(ab_1-a_1b)+(a_1{}^2+b_1{}^2+c_1{}^2)(ab_{11}-ba_{11})]\},$$

the term in [ ] is found to be $= -c_1\{a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)\}$, hence c_1 appears as a factor of the whole expression; and reducing the part independent of m, we find

$$dT = \frac{\sqrt{1-m^2}\,V^3 c_1\,dt}{\rho^2 - c'^2}\{(a_1b_{11} - a_{11}b_1) + m\,[a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\}.$$

Next, calculating the value of $adb - bda$, we have

$$a = \frac{V}{\rho}\{a_1 + m(ca_1 - c_1a)\},\ \ b = \frac{V}{\rho}\{b_1 - m(bc_1 - b_1c)\},$$

or, as these may be written,

$$a = \frac{V}{\rho}\{a_1(1 + cm) - ac_1m\},\ \ b = \frac{V}{\rho}\{b_1(1 + cm) - bc_1m\},$$

and we thence easily obtain

$$adb - bda = \frac{V^2dt}{\rho^2}(1 + cm)\{(a_1b_{11} - a_{11}b_1) + m\,[a_{11}(bc_1 - b_1c) + b_{11}(ca_1 - c_1a) + c_{11}(ab_1 - a_1b)]\},$$

viz. the factor in { } has the same value as in the expression for dT, and we thus have

$$\frac{dT}{adb - bda} = \frac{\sqrt{1-m^2}\,Vc_1\rho^2}{(1 + cm)(\rho^2 - c'^2)} = \frac{c\sqrt{1-m^2}}{\sqrt{1 - c^2m^2}\,(1 - c^2)},$$

that is,

$$dT = \frac{\sqrt{1-m^2}\,c\,(adb - bda)}{(1 - c^2)\sqrt{1 - c^2m^2}},$$

the required equation.

Secondly, for the term $\dfrac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}}$, we introduce Φ a function of θ, such that writing Φ' for the derived function we have

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4(1 - m^2)\Phi\Phi' + 4(1 - m^2)\theta\Phi'^2}},\ = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}}\ \text{suppose},$$

whence also

$$\sqrt{\theta + (m^2-1)\phi^2} = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}},\quad \frac{\phi}{\sqrt{\theta + (m^2-1)\phi^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}}.$$

Then writing

$$\sin\Theta = \frac{\Phi\sqrt{1-m^2}}{\sqrt{\theta}},\ \ \cos\Theta = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi\sqrt{1-m^2}}{\sqrt{\theta}},\ \ \cos\Theta_0 = \frac{\sqrt{\theta + (m^2-1)\phi^2}}{\sqrt{\theta}},$$

we find

$$\cos\Theta\,d\Theta = -\frac{\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\,\Phi^2}};$$

and similarly

$$\cos\Theta_0 d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\,\phi^2}}.$$

Hence

$$-d\Theta + d\Theta_0 = \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\,\Phi^2}} - \frac{\phi - 2\theta\phi'}{\sqrt{\theta + (m^2-1)\,\phi^2}}\right\} d\theta$$

$$= \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\phi - (\phi - 2\theta\phi')}{\sqrt{\theta + (m^2-1)\,\phi^2}}\right\} d\theta, \; = \frac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\,\phi^2}},$$

the required equation.

We find, moreover,

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{1-m^2}\sqrt{\theta + (m^2-1)\,\Phi^2}}{\sqrt{M}}, \quad \cos(\Theta - \Theta_0) = \frac{1 - 2(1-m^2)\,\Phi\Phi'}{\sqrt{M}},$$

which will be presently useful.

The differential equation now is $d\zeta - dT + d\Theta - d\Theta_0 = 0$, hence the integral equation (taking the constant of integration $=0$) is $\zeta = T - \Theta + \Theta_0$, or say

$$\sin\zeta = \sin(T - \Theta + \Theta_0),$$

viz. substituting for $\sin T$ and $\cos T$ their values, and observing that

$$\sin\zeta = \frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm}, \quad = \frac{1+cm}{\sqrt{\rho^2 - c'^2}}\,\frac{C+m}{1+Cm},$$

the factor $\dfrac{1}{\sqrt{\rho^2 - c'^2}}$ multiplies out, and we have

$$(1+cm)\frac{C+m}{1+Cm} = (c+m)\cos(\Theta - \Theta_0) - c''\sqrt{1-m^2}\sin(\Theta - \Theta_0).$$

And I further remark here that a former equation is

$$\Omega\,(1-m^2) = (1-c^2)(1+Cm)^2 - (1-c^2m^2)(C+m)^2,$$

that is,

$$\Omega\,\frac{1-m^2}{(1+Cm)^2} = (1-c^2)\left\{1 - \frac{(1-c^2m^2)(C+m)^2}{(1-c^2)(1+Cm)^2}\right\} = (1-c^2)\cos^2\zeta.$$

We thus have

$$\sqrt{\Omega} = \frac{1+Cm}{\sqrt{1-m^2}}\,\frac{\sqrt{\rho^2 - c'^2}}{\rho}\cos\zeta,$$

$$= \frac{1+Cm}{\rho\sqrt{1-m^2}}\{c''\sqrt{1-m^2}\cos(\Omega - \Omega_0) + (c+m)\sin(\Omega - \Omega_0)\}.$$

We have thus C, and consequently also A, B, x, y, z, all of them given as functions of t, θ; but the formulæ admit of further development.

Write

$$\mathfrak{s} = \frac{C+m}{1+Cm}, \text{ whence also } C = \frac{\mathfrak{s}-m}{1-m\mathfrak{s}}.$$

We have $C(1-m\mathfrak{s}) + m = \mathfrak{s}$, and hence $(1+\mathrm{c}m)\{C(1-m\mathfrak{s})+m\}$, $=(1+\mathrm{c}m)\mathfrak{s}$, $=(\mathrm{c}+m)\cos(\Theta-\Theta_0) - \mathrm{c}''\sqrt{1-m^2}\sin(\Theta-\Theta_0)$. Using the value of C given by this equation, and calculating from it those of A, B; then writing for shortness

$$\begin{aligned} X &= \mathrm{a}\sqrt{1-m^2}\cos(\Theta-\Theta_0) - (\mathrm{a}''-m\mathrm{b}')\sin(\Theta-\Theta_0),\\ Y &= \mathrm{b}\sqrt{1-m^2}\cos(\Theta-\Theta_0) - (\mathrm{b}''+m\mathrm{a}')\sin(\Theta-\Theta_0),\\ Z &= (\mathrm{c}+m)\cos(\Theta-\Theta_0) - \mathrm{c}''\sqrt{1-m^2}\sin(\Theta-\Theta_0), \end{aligned}$$

we have

$$\begin{aligned} A(1-m\mathfrak{s})(1+\mathrm{c}m) &= \sqrt{1-m^2}X,\\ B(1-m\mathfrak{s})(1+\mathrm{c}m) &= \sqrt{1-m^2}Y,\\ C(1-m\mathfrak{s})(1+\mathrm{c}m) &= Z - m(1+\mathrm{c}m), \end{aligned}$$

to which I join

$$\mathfrak{s}(1+\mathrm{c}m) = Z.$$

By way of verification, observe that $A^2+B^2+C^2=1$, and that the equations give

$$(1-m\mathfrak{s})^2(1+\mathrm{c}m)^2 = (1-m^2)(X^2+Y^2+Z^2) + m^2Z^2 - 2mZ(1+\mathrm{c}m) + m^2(1+\mathrm{c}m)^2;$$

we have

$$X^2+Y^2+Z^2 = (1+\mathrm{c}m)^2, \quad Z = \mathfrak{s}(1+\mathrm{c}m),$$

and hence the identity

$$(1-m\mathfrak{s})^2(1+\mathrm{c}m)^2 = (1-m^2+m^2\mathfrak{s}^2-2m\mathfrak{s}+m^2)(1+\mathrm{c}m)^2.$$

Proceeding to calculate the values of x, y, z, recollecting that

$$\sqrt{1-c^2m^2} = \frac{1}{\rho}(1+\mathrm{c}m),$$

we have

$$\begin{aligned} x(1+\mathrm{c}m) &= A\phi(1+\mathrm{c}m) + \rho(bC-cB)\sqrt{\theta+(m^2-1)\phi^2},\\ &= A\phi(1+\mathrm{c}m) + \{(\mathrm{b}'-m\mathrm{a}'')C - \mathrm{c}'B\}\sqrt{\theta+(m^2-1)\phi^2}, \end{aligned}$$

that is,

$$\begin{aligned} x(1+\mathrm{c}m)(1-m\mathfrak{s}) &= \phi\sqrt{1-m^2}X + \frac{1}{1+\mathrm{c}m}\{(\mathrm{b}'-m\mathrm{a}'')(Z-m(1+\mathrm{c}m))\\ &\qquad - \mathrm{c}'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &= \phi\sqrt{1-m^2}X + \frac{1}{1+\mathrm{c}m}\{(\mathrm{b}'-m\mathrm{a}'')Z - \mathrm{c}'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &\qquad - m(\mathrm{b}'-m\mathrm{a}'')\sqrt{\theta+(m^2-1)\phi^2}, \end{aligned}$$

where the term $(\mathrm{b}' - m\mathrm{a}'')\,Z - \mathrm{c}'\sqrt{1-m^2}\,Y$ contains the factor $1 + \mathrm{c}m$; in fact, this is

$$= (\mathrm{b}' - m\mathrm{a}'')\{(\mathrm{c} + m)\cos(\Theta - \Theta_0) - \mathrm{c}''\sqrt{1-m^2}\sin(\Theta - \Theta_0)\}$$
$$- \mathrm{c}'\sqrt{1-m^2}\{\mathrm{b}\sqrt{1-m^2}\cos(\Theta - \Theta_0) - (\mathrm{b}'' + m\mathrm{a}')\sin(\Theta - \Theta_0)\}.$$

The coefficient of the cosine is $(\mathrm{b}' - m\mathrm{a}'')(\mathrm{c} + m) - \mathrm{bc}'(1 - m^2)$, which is

$$= \mathrm{b}'\mathrm{c} - \mathrm{bc}' + m(\mathrm{b}' - \mathrm{ca}'') + m^2(-\mathrm{a}'' + \mathrm{bc}'),\ = -\mathrm{a}'' + m(\mathrm{b}' - \mathrm{ca}'') + m^2(-\mathrm{b}'\mathrm{c}),$$
$$= (1 + \mathrm{c}m)(-\mathrm{a}'' + m\mathrm{b}'),$$

and similarly the coefficient of $\sqrt{1-m^2}$, multiplied by the sine, is

$$-\mathrm{c}''(\mathrm{b}' - m\mathrm{a}'') + \mathrm{c}'(\mathrm{b}'' + m\mathrm{a}'),\ = -\mathrm{b}'\mathrm{c}'' + \mathrm{b}''\mathrm{c}' + m(\mathrm{a}'\mathrm{c}' + \mathrm{a}''\mathrm{c}''),$$
$$= -\mathrm{a} + m(-\mathrm{ac}),\ = (1 + \mathrm{c}m)(-\mathrm{a}).$$

Calculating in like manner the values of y and z, and putting for shortness

$$X_1 = (-\mathrm{a}'' + m\mathrm{b}')\cos(\Theta - \Theta_0) - \mathrm{a}\sqrt{1-m^2}\sin(\Theta - \Theta_0),$$
$$Y_1 = (-\mathrm{b}'' - m\mathrm{a}')\cos(\Theta - \Theta_0) - \mathrm{b}\sqrt{1-m^2}\sin(\Theta - \Theta_0),$$
$$Z_1 = (-\mathrm{c}''\sqrt{1-m^2})\cos(\Theta - \Theta_0) + (\mathrm{c} + m)\sin(\Theta - \Theta_0),$$

we have

$$x = \phi\sqrt{1-m^2}X + X_1\sqrt{\theta + (m^2-1)\phi^2} - m(\mathrm{b}' - m\mathrm{a}'')\sqrt{\theta + (m^2-1)\phi^2},$$
$$y = \phi\sqrt{1-m^2}Y + Y_1\sqrt{\theta + (m^2-1)\phi^2} + m(\mathrm{a}' + m\mathrm{b}'')\sqrt{\theta + (m^2-1)\phi^2},$$
$$z = \sqrt{1-m^2}\{\phi\sqrt{1-m^2}Z + Z_1\sqrt{\theta + (m^2-1)\phi^2}\},$$

which are the required expressions of x, y, z in terms of t and θ. It will be noticed that X, X_1, Y, Y_1, Z, Z_1, each contain a term with $\cos(\Theta - \Theta_0)$ and one with $\sin(\Theta - \Theta_0)$; but as the terms in X_1, Y_1, Z_1 are each multiplied by $\sqrt{\theta + (m^2-1)\phi^2}$, the cosine and sine terms of X, X_1, of Y, Y_1 and of Z, Z_1 do not in any case unite into a single term.

I remark that we have identically

$$aX + bY + c\sqrt{1-m^2}Z = 0,$$
$$aX_1 + bY_1 + c\sqrt{1-m^2}Z_1 = 0.$$

The foregoing values of x, y, z thus satisfy $ax + by + cz = 0$, which is one of the six equations. The others of them might be verified without difficulty. I recall that we have a, b, $c = \frac{1}{\rho}(\mathrm{a}' + m\mathrm{b}'')$, $\frac{1}{\rho}(\mathrm{b}' - m\mathrm{a}'')$, $\frac{1}{\rho}\mathrm{c}'$; the six equations might therefore be written

$$A^2 + B^2 + C^2 = 1,$$
$$(\mathrm{a}' + m\mathrm{b}'')\,x + (\mathrm{b}' - m\mathrm{a}'')\,y + \mathrm{c}'z = 0,$$
$$(\mathrm{a}' + m\mathrm{b}'')\,A + (\mathrm{b}' - m\mathrm{a}'')\,B + \mathrm{c}'C = -\mathrm{c}'m,$$
$$x^2 + y^2 + (z - m\phi)^2 = \theta + m^2\phi^2,$$
$$Ax + By + C(z - m\phi) = \phi,$$
$$Adx + Bdy + Cdz = 0.$$

The Case PS, 1° = Serret's First Case of PS.

This is at once deduced from *PS*, 3° by writing therein $m = 0$; the formulæ are a good deal more simple. We introduce, as before, the rectangular coefficients a, b, c, a′, b′, c′, a″, b″, c″; and the values of a, b, c then are a′, b′, c′. The six equations, using therein these values for a, b, c, are

$$\begin{aligned} A^2 + B^2 \ \ + C^2 &= 1, \\ \mathrm{a}'x + \mathrm{b}'y \ + \mathrm{c}'z &= 0, \\ \mathrm{a}'A + \mathrm{b}'B + \mathrm{c}'C &= 0, \\ x^2 + y^2 \ \ + z^2 &= \theta, \\ Ax + By \ + Cz &= \phi, \\ Adx + Bdy + Cdz &= 0. \end{aligned}$$

The function Φ is such that

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4\Phi\Phi' + 4\theta\Phi'^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}}.$$

We have

$$\sin\Theta = \frac{\Phi}{\sqrt{\theta}} \cos\Theta = \frac{\sqrt{\theta - \Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi}{\sqrt{\theta}} \cos\Theta_0 = \frac{\sqrt{\theta - \phi^2}}{\sqrt{\theta}},$$

and thence

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{\theta - \Phi^2}}{\sqrt{M}};\ \ \cos(\Theta - \Theta_0) = \frac{1 - 2\Phi\Phi'}{\sqrt{M}}.$$

Also

$$\sin\zeta = \frac{C}{\sqrt{1 - \mathrm{c}^2}},\quad \cos\zeta = \frac{\sqrt{1 - \mathrm{c}^2 - C^2}}{\sqrt{1 - \mathrm{c}^2}};\quad \sin T = \frac{\mathrm{c}}{\sqrt{1 - \mathrm{c}'^2}},\quad \cos T = \frac{\mathrm{c}''}{\sqrt{1 - \mathrm{c}^2}},$$

$$\zeta = T - \Theta + \Theta_0,\quad C = \mathrm{c}\cos(\Theta - \Theta_0) - \mathrm{c}''\sin(\Theta - \Theta_0),$$

$$\sqrt{1 - \mathrm{c}^2 - C^2} = \mathrm{c}''\cos(\Theta - \Theta_0) + \mathrm{c}\sin(\Theta - \Theta_0).$$

We have

$$\begin{aligned} A = X &= \mathrm{a}\cos(\Theta - \Theta_0) - \mathrm{a}''\sin(\Theta - \Theta_0);\quad X_1 = \mathrm{a}''\cos(\Theta - \Theta_0) + \mathrm{a}\sin(\Theta - \Theta_0), \\ B = Y &= \mathrm{b}\cos(\Theta - \Theta_0) - \mathrm{b}''\sin(\Theta - \Theta_0);\quad Y_1 = \mathrm{b}''\cos(\Theta - \Theta_0) + \mathrm{b}\sin(\Theta - \Theta_0), \\ C = Z &= \mathrm{c}\cos(\Theta - \Theta_0) - \mathrm{c}''\sin(\Theta - \Theta_0);\quad Z_1 = \mathrm{c}''\cos(\Theta - \Theta_0) + \mathrm{c}\sin(\Theta - \Theta_0), \end{aligned}$$

and then

$$\begin{aligned} x &= X\phi + X_1\sqrt{\theta - \phi^2}, \\ y &= Y\phi + Y_1\sqrt{\theta - \phi^2}, \\ z &= Z\phi + Z_1\sqrt{\theta - \phi^2}, \end{aligned}$$

which are the expressions of the coordinates in terms of the parameters t and θ.

I consider next the case

PS, 4^0 = *Serret's Third Case of PS.*

The six equations are

$$\begin{aligned} &A^2 + B^2 + C^2 &&= 1\ ,\\ &ax + by &&= lm,\\ &Aa + Bb &&= l\ ,\\ &x^2 + y^2 + z^2 - 2\theta z &&= 2v,\\ &Ax + By + C(z-\theta) &&= m\ ,\\ &Adx + Bdy + Cdz &&= 0\ ; \end{aligned}$$

where θ has been written in place of γ: m is a given constant; a, b, l are functions of t; v is a function of θ. The equation $ax + by = lm$ evidently denotes that the planes of the plane curves of curvature are all of them parallel to the axis of z, or, what is the same thing, they envelope a cylinder; in the particular case $m = 0$, they all of them pass through the axis of z. In the general case, the required surface is the parallel surface, at the normal distance m, to the surface which belongs to the particular case $m = 0$. This is not assumed in the investigation which follows; but it will be readily perceived how the theorem is involved in, and in fact proved by, the investigation.

I obtain the solution synthetically as follows:

Taking T, a, b functions of t, $\mathrm{a}^2 + \mathrm{b}^2 = 1$; T_1, a_1, b_1 their derived functions, $\mathrm{aa}_1 + \mathrm{bb}_1 = 0$; $\Omega = \dfrac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2$; Θ a function of θ, Θ' its derived function,

$$M = \frac{2\Theta}{\Theta'};\ P = \frac{2\sqrt{T\Theta}}{T+\Theta},\ Q = \frac{T-\Theta}{T+\Theta},$$

and therefore $P^2 + Q^2 = 1$; then writing

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{\Omega}}\left(-\mathrm{b}\frac{T_1}{2T} + \mathrm{b}_1 Q\right),\\ B_0 &= \frac{1}{\sqrt{\Omega}}\left(\ \mathrm{a}\frac{T_1}{2T} - \mathrm{a}_1 Q\right),\\ C_0 &= \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P, \end{aligned}$$

where $A_0^2 + B_0^2 + C_0^2 = 1$, we assume

$$\begin{aligned} x &= mA_0 + \mathrm{a}MP,\\ y &= mB_0 + \mathrm{b}MP,\\ z &= mC_0 + \theta + MQ, \end{aligned}$$

equations which determine x, y, z as functions of the parameters t and θ. As will presently be shown, we have $A_0dx+B_0dy+C_0dz=0$; and we have thus $A, B, C=A_0, B_0, C_0$; and this being so, we easily verify the six equations

$$\begin{aligned}
A^2 \;\; &+ B^2 + C^2 &&= 1,\\
\mathrm{b}x \;\; &- \mathrm{a}y &&= m\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right),\\
\mathrm{b}A \;\; &- \mathrm{a}B &&= \;\;\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right),\\
x^2 \;\; &+ y^2 + (z-\theta)^2 &&= m^2 + M^2 + \theta^2,\\
Ax \;\; &+ By + C(z-\theta) = m, &&\\
Adx \;\; &+ Bdy + Cdz &&= 0,
\end{aligned}$$

which are the six equations of the problem with the values $a=\mathrm{b}$, $b=-\mathrm{a}$, $l=-\frac{2T_1}{T}\frac{1}{\sqrt{\Omega}}$, $2v=m^2+M^2$, for a, b, l and $2v$.

We in fact at once obtain the third equation $\mathrm{b}A_0-\mathrm{a}B_0=-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}$, and thence the second equation $\mathrm{b}x-\mathrm{a}y=m(\mathrm{b}A_0-\mathrm{a}B_0)$, $=m\left(-\frac{T_1}{2T}\frac{1}{\sqrt{\Omega}}\right)$; then for the fifth equation, we have

$$A_0x+B_0y+C_0(z-\theta)=m+M\{(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q\},\ =m,$$

since $(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q=0$; and for the fourth equation, we have

$$x^2+y^2+(z-\theta)^2=m^2+2m\{M(A_0\mathrm{a}+B_0\mathrm{b})P+C_0Q\}+M^2,\ =m^2+M^2.$$

It remains only to prove the assumed equation $A_0dx+B_0dy+C_0dz=0$. Writing for a moment $X, Y, Z=\mathrm{a}MP$, $\mathrm{b}MP$, $\theta+MQ$, we have

$$\begin{aligned}
A_0dx+B_0dy+C_0dz &= A_0(mdA_0+dX)+B_0(mdB_0+dY)+C_0(mdC_0+dZ),\\
&= A_0dX+B_0dY+C_0dZ,
\end{aligned}$$

since $A_0dA_0+B_0dB_0+C_0dC_0=0$ in virtue of $A_0^2+B_0^2+C_0^2=1$.

We have thus to show that, if $X, Y, Z=\mathrm{a}MP$, $\mathrm{b}MP$, $\theta+MQ$, then

$$A_0dX+B_0dY+C_0dZ=0;$$

say we have

$$\begin{aligned}
dX &= pdt+p'd\theta,\\
dY &= qdt+q'd\theta,\\
dZ &= rdt+r'd\theta,
\end{aligned}$$

then the required values of A_0, B_0, C_0 are proportional to $qr'-q'r$, $rp'-r'p$, $pq'-p'q$, and the sum of their squares is $=1$. Writing for shortness $MP=R$, $MQ=S$, we have

$$\begin{aligned}
p &= \mathrm{a}_1R+\mathrm{a}R_1, & p' &= \mathrm{a}R',\\
q &= \mathrm{b}_1R+\mathrm{b}R_1, & q' &= \mathrm{b}R',\\
r &= S_1, & r' &= 1+S';
\end{aligned}$$

hence

$$\begin{aligned} qr' - q'r &= \ \ \mathrm{b}\,[R_1(1+S') - R'S_1] + \mathrm{b}_1 R(1+S'),\\ rp' - r'p &= -\,\mathrm{a}\,[R_1(1+S') - R'S_1] - \mathrm{a}_1 R(1+S'),\\ pq' - p'q &= -\,(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,RR'. \end{aligned}$$

Here

$$R' = MP' + M'P, \quad R_1 = MP_1,$$
$$S' = MQ' + M'Q, \quad S_1 = MQ_1,$$

and hence

$$\begin{aligned} RS' - R'S &= M^2(PQ' - P'Q),\\ R_1S' - R'S_1 &= M^2(P_1Q' - P'Q_1) + MM'(P_1Q - PQ_1); \end{aligned}$$

moreover, from the values of P and Q, we have

$$P' = \ \frac{\Theta'}{2\Theta}PQ, \ = \ \frac{PQ}{M}, \quad P_1 = -\frac{T_1}{2T}PQ,$$
$$Q' = -\frac{\Theta'}{2\Theta}P^2 \ = -\frac{P^2}{M}, \quad Q_1 = \ \frac{T_1}{2T}P^2,$$

and thence

$$P_1Q - PQ_1 = -\frac{T_1P}{2T}; \ PQ' - P'Q = -\frac{P}{M}; \ P_1Q' - P'Q_1 = 0;$$

also

$$R' = \ PQ + M'P, \ = P(Q+M'), \qquad 1 + S = 1 - P^2 + M'Q, \ = Q(Q+M');$$
$$R_1 = -\frac{T_1}{2T}MPQ\ , \qquad RS' - R'S = -PM, \qquad R_1S' - R'S_1 = -\frac{T_1}{2T}MPM',$$

and consequently

$$\begin{aligned} R_1(1+S') - R'S_1 &= -\frac{T_1}{2T}MP\ \ (Q+M'),\\ R\ (1+S') &= \quad MPQ\,(Q+M'),\\ RR' &= \quad MP^2\ (Q+M'). \end{aligned}$$

Hence the foregoing expressions for $qr' - q'r$, $rp' - r'p$, $pq' - p'q$ each contain the factor $MP(Q+M')$; omitting this factor, the expressions are

$$\left\{-\,\mathrm{b}\,\frac{T_1}{2T} + \mathrm{b}_1 Q\right\}, \quad \left\{\mathrm{a}\,\frac{T_1}{2T} - \mathrm{a}_1 Q\right\}, \ -(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P;$$

the sum of the squares of these values is $= \frac{T_1^2}{4T^2} + \mathrm{a}_1^2 + \mathrm{b}_1^2$, $= \Omega$, and we have thus the required values

$$A_0 = \frac{1}{\sqrt{\Omega}}\left\{-\,\mathrm{b}\,\frac{T_1}{2T} + \mathrm{b}_1 Q\right\},$$
$$B_0 = \frac{1}{\sqrt{\Omega}}\left\{\ \mathrm{a}\,\frac{T_1}{2T} - \mathrm{a}_1 Q\right\},$$
$$C_0 = \frac{-1}{\sqrt{\Omega}}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})\,P,$$

which completes the proof.

In the case $m=0$, the solution is

$$x,\ y,\ z=\frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad \frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad \theta+\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}.$$

Bonnet, in the paper (*Jour. Ecole Polyt.* t. XX.) referred to at the beginning of this memoir, gives for this case (see p. 199) a solution which he says is equivalent to that obtained by Joachimsthal in the paper "Demonstrationes theorematum ad superficies curvas spectantium," *Crelle,* t. XXX. (1846), pp. 347—350; viz. Joachimsthal's form is

$$x=\frac{\mu\sin L\sin\lambda}{1+\cos L\cos M},$$

$$y=\frac{\mu\sin L\cos\lambda}{1+\cos L\cos M},$$

$$z=\frac{\mu\cos L\sin M}{1+\cos L\cos M}+\int\cot M\,d\mu,$$

where L, M denote arbitrary functions of the parameters λ, μ respectively. To identify these with the foregoing form, I write

$$\sin\lambda=-\mathrm{a},\quad \cos L=-\frac{T-1}{T+1},\quad \cos M=\frac{\Theta-1}{\Theta+1},\quad \mu=\frac{4\Theta\sqrt{\Theta}}{\Theta'(\Theta+1)};$$

$$\cos\lambda=-\mathrm{b},\quad \sin L=\frac{-2\sqrt{T}}{T+1},\quad \sin M=\frac{-2\sqrt{\Theta}}{\Theta+1};$$

we thus have

$$\frac{\sin L\sin\lambda}{1+\cos L\cos M}=\mathrm{a}\,\frac{2\sqrt{T}}{T+1}\div 1-\frac{(T-1)(\Theta-1)}{(T+1)(\Theta+1)},\quad =\frac{\mathrm{a}\sqrt{T}(\Theta+1)}{T+\Theta},$$

and thence

$$x=\frac{2\Theta}{\Theta'}\,\mathrm{a}\,\frac{2\sqrt{T\Theta}}{T+\Theta},\quad y=\frac{2\Theta}{\Theta'}\,\mathrm{b}\,\frac{2\sqrt{T\Theta}}{T+\Theta}.$$

Moreover,

$$\frac{\cos L\sin M}{1+\cos L\cos M}=\frac{\sqrt{\Theta}(T-1)}{T+\Theta},$$

and thence the first term of z is $=\dfrac{2\Theta}{\Theta'}\,\dfrac{2\Theta(T-1)}{(\Theta+1)(T+\Theta)}$; or observing that

$$2\Theta(T-1)=(\Theta+1)(T-\Theta)+(\Theta-1)(T+\Theta),$$

this is

$$=\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}+\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

or we have

$$z=\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta}+\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)}+\int\cot M\,d\mu.$$

Here

$$\cot M\,d\mu = -\frac{\Theta-1}{2\sqrt{\Theta}}\,\frac{4\Theta\sqrt{\Theta}}{\Theta(\Theta+1)}\left\{\frac{\frac{3}{2}\Theta'}{\Theta}-\frac{\Theta'}{\Theta+1}-\frac{\Theta''}{\Theta'}\right\}d\theta$$

$$=\frac{\Theta(\Theta-1)}{\Theta+1}\left\{-\frac{3}{\Theta}+\frac{2}{\Theta+1}+\frac{2\Theta''}{\Theta'^2}\right\}d\theta.$$

But writing

$$\xi=\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)},$$

we have

$$d\xi=\frac{2\Theta(\Theta-1)}{\Theta'(\Theta+1)}\left\{\frac{\Theta'}{\Theta}+\frac{\Theta'}{\Theta-1}-\frac{\Theta'}{\Theta+1}-\frac{\Theta''}{\Theta'}\right\}d\theta$$

$$=\frac{\Theta(\Theta-1)}{\Theta+1}\left\{\frac{2}{\Theta}+\frac{2}{\Theta-1}-\frac{2}{\Theta+1}-\frac{2\Theta''}{\Theta'^2}\right\}d\theta,$$

and thence

$$d\xi+\cot M\,d\mu=\frac{\Theta(\Theta-1)}{\Theta+1}\left\{-\frac{1}{\Theta}+\frac{2}{\Theta-1}\right\}d\theta,\ =d\theta,$$

and consequently

$$\xi+\int\cot M\,d\mu=\theta,$$

and the value of z thus is

$$z=\theta+\frac{2\Theta}{\Theta'}\,\frac{T-\Theta}{T+\Theta},$$

which completes the identification.

Bonnet's formulæ just referred to, making a slight change of notation and correcting a sign, are

$$x=\frac{\Gamma'\sin\theta}{\cos i(c+\Theta)},$$

$$y=\frac{\Gamma'\cos\theta}{\cos i(c+\Theta)},$$

$$z=\Gamma+i\Gamma'\tan i(c+\Theta),$$

where Γ, Θ are arbitrary functions of the parameters c, θ respectively. To identify these with Joachimsthal's, write

$$\sin\lambda=\sin\theta,\quad \cos M=i\cot ic,\quad \cos L=i\cot i\Theta,\quad \mu=-\Gamma'\operatorname{cosec} ic,$$

$$\cos\lambda=\cos\theta,\quad \sin M=\operatorname{cosec} ic,\quad \sin L=\operatorname{cosec} i\Theta,$$

$$\cot M=i\cos ic,\quad \cot L=i\cos i\Theta;$$

we have

$$x=\frac{\mu\operatorname{cosec} i\Theta\sin\theta}{1-\cot ic\cot i\Theta}=\frac{-\mu\sin ic\sin\theta}{\cos i(c+\Theta)},\ =\frac{\Gamma'\sin\theta}{\cos i(c+\Theta)};$$

and similarly

$$y=\frac{\Gamma'\cos\theta}{\cos i(c+\Theta)}.$$

Moreover, the first term of z is

$$\frac{\mu i \cot i\Theta \operatorname{cosec} ic}{1-\cot ic \cot i\Theta} = \frac{-\mu i \cos i\Theta}{\cos i(c+\Theta)} = \frac{i\Gamma' \cos i\Theta}{\sin ic \cos i(c+\Theta)};$$

or since $i\Theta = i(c+\Theta) - ic$, and thence

$$\cos i\Theta = \cos ic \cos i(c+\Theta) + \sin ic \sin i(c+\Theta),$$

this is

$$= i\Gamma' \{\cot ic + \tan i(c+\Theta)\},$$

and we have

$$z = i\Gamma' \tan i(c+\Theta) + i\Gamma' \cot ic + \int \cot M\, d\mu.$$

But from the equation $\mu = -\Gamma' \operatorname{cosec} ic$, we obtain

$$d\mu = (-\Gamma'' \operatorname{cosec} ic + i\Gamma' \operatorname{cosec} ic \cot ic)\, dc;$$

whence

$$\cot M\, d\mu = (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc,$$

and thence

$$d\left(i\Gamma' \cot ic + \int \cot M\, d\mu\right) = (i\Gamma'' \cot ic + \Gamma' \operatorname{cosec}^2 ic)\, dc$$
$$+ (-i\Gamma'' \cot ic - \Gamma' \cot^2 ic)\, dc, \; = \Gamma' dc;$$

that is,

$$i\Gamma' \cot ic + \int \cot M\, d\mu = \Gamma,$$

and consequently

$$z = \Gamma + i\Gamma' \tan i(c+\Theta),$$

which completes the identification of Bonnet's formula with Joachimsthal's.

SS, The Sets of Curves of Curvature each Spherical.

The six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&= 1,\\
&x^2+y^2+z^2-2ax-2by-2cz-2u &&= 0,\\
&A(x-a)+B(y-b)+C(z-c)-l &&= 0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2\upsilon &&= 0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\gamma)-\lambda &&= 0,\\
&A\,dx+B\,dy+C\,dz &&= 0;
\end{aligned}$$

the condition being

$$a\alpha + b\beta + c\gamma - l\lambda + u + \upsilon = 0.$$

The cases are

	a	b	c	l	u	α	β	γ	λ	v
SS, 1°	0	0	0	l	$\frac{1}{2}(ml+m')$	α	β	γ	$\frac{1}{2}m$	$-\frac{1}{2}m'$
SS, 2°	0	0	c	l	$\frac{1}{2}(ml+m')$	α	β	0	$\frac{1}{2}m$	$-\frac{1}{2}m'$
SS, 3°	0	0	c	$mc+\frac{1}{2}m'$	$-\frac{1}{2}m''c-m'''$	α	β	$m\lambda+\frac{1}{2}m''$	λ	$\frac{1}{2}m'\lambda+m'''$
SS, 4°	0	b	c	$mc+\ m'$	$mm''c+m'm''-m'''$	α	0	γ	$\frac{1}{m}\gamma+m''$	$\frac{m'}{m}\gamma+m'''$,

where m, m', m'', m''' are constants; b, c, l functions of t; α, β, γ, λ functions of θ.

SS, 1° gives circles (i.e. the curves of curvature of one set are circles).

SS, 2° is Serret's first case of SS.

SS, 3° gives circles.

SS, 4° is Serret's second case of SS.

SS, 2° = Serret's First Case of SS.

Writing for convenience $m'=-f^2$, the six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&x^2+y^2+z^2-2cz-ml+f^2 &&=0,\\
&Ax+By+C(z-c)-l &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-f^2 &&=0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\lambda) &&=0,\\
&Adx+Bdy+Cdz &&=0,
\end{aligned}$$

where m, f are constants; c, l are functions of t; α, β, λ functions of θ. The first set of spheres have no points in common, but the second set have in common the two points $x=0$, $y=0$, $z=\pm f$. Hence inverting (by reciprocal radius vectors) with one of these points, say $(0, 0, f)$ as centre, the spheres of the first set will continue spheres, but the spheres of the second set will be changed into planes, and the required surface is thus the inversion of a surface PS, which is in fact PS, 3: say this surface PS is the "Inversion" of SS. We invert by the formulæ

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where $\Omega=X^2+Y^2+(Z-f)^2$.

Writing the equation for the second set of spheres in the form

$$x^2+y^2+(z-f)^2-2\alpha x-2\beta y+2f(z-f)=0,$$

the transformed equation is at once found to be

$$-2\alpha X-2\beta Y+2f(Z-f)+K^2=0,$$

or say

$$\alpha X+\beta Y-fZ+f^2-\tfrac{1}{2}K^2=0;$$

viz. this gives the planes of the Inversion.

Similarly for the first set of spheres, writing the equation in the form

$$x^2+y^2+(z-f)^2+2(f-c)(z-f)+2f(f-c)-ml=0,$$

the transformed equation is found to be

$$\{2f(f-c)-ml\}\{X^2+Y^2+(Z-f)^2\}+2(f-c)K^2(Z-f)+K^4=0;$$

viz. this is

$$\{2f(f-c)-ml\}(X^2+Y^2+Z^2)+2Z\{(f-c)(-2f^2+K^2)+fml\}$$
$$+\{2f(f-c)(f^2-K^2)-f^2ml+K^4\}=0,$$

which gives the spheres of the Inversion. The two equations take a more simple form if we write therein $K^2=2f^2$; viz. they then become

$$\alpha X+\beta Y-fZ=0,$$
$$(2f^2-2fc-ml)(X^2+Y^2+Z^2)+2Zfml+f^2(2f^2+2cf-ml)=0;$$

or, say these are

$$\left\{\begin{array}{l}\dfrac{\alpha X+\beta Y-fZ}{\sqrt{\alpha^2+\beta^2+f^2}}=0,\\[2ex] X^2+Y^2+Z^2+\dfrac{2fml}{2f^2-2fc-ml}Z+\dfrac{f^2(2f^2+2cf-ml)}{2f^2-2fc-ml}=0.\end{array}\right.$$

Interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY+cZ=0,$$
$$X^2+Y^2+Z^2-2\gamma Z-2v=0,$$

where $a^2+b^2+c^2=1$; and the Inversion is thus a surface PS, 3^0.

SS, 4^0 = *Serret's Second Case of SS.*

Writing for convenience $m'=-mf$, $m'''=\tfrac{1}{2}(e^2+f^2)$, $mm''=-g$, and therefore $m'm''=fg$, the six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&x^2+y^2+z^2-2by-2c(z-g)-2fg+e^2+f^2 &&=0,\\
&Ax+B(y-b)+C(z-c)+m(f-c) &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\gamma(z-f)-e^2-f^2 &&=0,\\
&A(x-\alpha)+By+C(z-\gamma)-\frac{1}{m}(g+\gamma) &&=0,\\
&Adx+Bdy+Cdz &&=0,
\end{aligned}$$

where e, f, g, m are constants; b, c are functions of t; α, γ functions of θ.

The spheres of the first set pass all of them through the two points

$$x = \pm\sqrt{2fg - e^2 - f^2 - g^2},\quad y = 0,\qquad z = g,$$

and those of the second set pass all of them through the two points

$$x' = 0,\qquad\qquad y' = \pm e,\quad z' = f,$$

where observe that these are such that

$$(x - x')^2 + (y - y')^2 + (z - z')^2 = 0;$$

viz. the distance of each point of the first pair from each point of the second pair is $=0$. The pairs of points are one real, the other imaginary: but this is quite consistent with the reality of the spheres.

The first pair of points lies in a line parallel to the axis of x, meeting the axis of z at the point $z = g$; and the second pair in a line parallel to the axis of y, cutting the axis of z at the point $z = f$. It is clear that we can, without loss of generality, by moving the origin along the axis of z, in effect make g to be $= -f$; the equations of the two sets of spheres thus become

$$x^2 + y^2 + z^2 - 2by - 2c\,(z + f) + e^2 + 3f^2 = 0,$$
$$x^2 + y^2 + z^2 - 2ax - 2\gamma\,(z - f) - e^2 - f^2 = 0,$$

or, if in these equations for e^2 we write $e^2 - 2f^2$, the equations become

$$x^2 + y^2 + z^2 - 2by - 2c\,(z + f) + e^2 + f^2 = 0,$$
$$x^2 + y^2 + z^2 - 2ax - 2\gamma\,(z - f) - e^2 + f^2 = 0,$$

which are very symmetrical forms.

The spheres of the first set pass through the two points

$$\pm\sqrt{-e^2 - 2f^2},\qquad 0\quad,\ -f,$$

and those of the second set through the two points

$$0\qquad,\ \pm\sqrt{e^2 - 2f^2},\quad f,$$

where, of course, the two pairs of points are related as is mentioned above.

By taking as centre of inversion a point of the first pair, we invert the first set of spheres into planes and the second set into spheres; and similarly, by taking a point of the second pair, we invert the first set of spheres into spheres and the second set into planes. By reason of the symmetry of the system, it is quite indifferent which point is chosen; and taking it to be a point of the second pair, and writing for convenience $n = \sqrt{e^2 - 2f^2}$ (n is, in fact, the quantity originally denoted by e), then the points of the first pair are

$$\pm\sqrt{-n^2 - 4f^2},\quad 0,\ -f,$$
$$0\qquad,\ \pm n,\quad f,$$

and I take for centre of inversion the point $(0, n, f)$.

Observe that, if $e=0$, $f=0$, then the four points coincide at the origin, and taking this as centre of inversion, the two sets of spheres are each changed into planes, and the Inversion of the surface SS is thus a surface PP; this particular case will be considered further on, but I first consider the general case.

The formulæ of inversion are

$$x=\frac{K^2X}{\Omega},\quad y-n=\frac{K^2(Y-n)}{\Omega},\quad z-f=\frac{K^2(Z-f)}{\Omega},$$

where

$$\Omega=X^2+(Y-n)^2+(Z-f)^2.$$

Writing the equation of the second set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2-2\alpha x+2n(y-n)+2(f-\gamma)(z-f)=0,$$

the transformed equation is

$$-\alpha X+n(Y-n)+(f-\gamma)(Z-f)+\tfrac{1}{2}K^2=0,$$

which gives the planes of the Inversion.

Similarly, writing the equation of the first set of spheres in the form

$$x^2+(y-n)^2+(z-f)^2+2(n-b)(y-n)+2(f-c)(z-f)+2n^2-2bn+4f(f-c)=0,$$

the transformed equation is

$$\{n^2-bn+2f(f-c)\}\{X^2+(Y-n)^2+(Z-f)^2\}$$
$$+K^2\{(n-b)(Y-n)+(f-c)(Z-f)\}+\tfrac{1}{2}K^4=0,$$

which gives the spheres of the Inversion.

Changing the origin, the two equations may be written

$$-\alpha X+nY+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$\{n^2-bn+2f(f-c)\}(X^2+Y^2+Z^2)+K^2\{(n-b)Y+(f-c)Z\}+\tfrac{1}{2}K^4=0.$$

I stop to consider a particular case. Suppose $n=0$; the equations are

$$-\alpha X+(f-\gamma)Z+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Y+\frac{K^2}{2f}Z+\frac{K^4}{4f(f-c)}=0,$$

or, interchanging herein Y and Z, they are

$$-\alpha X+(f-\gamma)Y+\tfrac{1}{2}K^2=0,$$
$$X^2+Y^2+Z^2+\frac{K^2}{2f}Y-\frac{bK^2}{2f(f-c)}Z+\frac{K^4}{4f(f-c)}=0;$$

and if for Y we write $Y-\frac{K^2}{4f}$, then the equations become

$$-\alpha X+(f-\gamma)Y+K^2\frac{f+\gamma}{4f}=0,$$

$$X^2+Y^2+Z^2-\frac{bK^2}{2f(f-c)}Z+\frac{K^4(5f-4c)}{f^2(f-c)}=0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY=lm,$$

$$X^2+Y^2+Z^2-2\theta Z=2v,$$

which belong to PS, 4°. Hence, in this particular case, $n=0$; the Inversion is PS, 4°.

Reverting to the general case, and to the two equations obtained above, observe that, in the second of the two equations, the terms in Y, Z have the variable coefficients $n-b$ and $f-c$, so that it does not at first sight seem as if these terms could by a transformation of coordinates be reduced to a single term.

But if, again changing the origin, we write $Y-\frac{\frac{1}{2}K^2}{n}$ for Y, the two equations become

$$-\alpha X+nY+(f-\gamma)Z=0,$$

$$\{n^2-bn+2f(f-c)\}(X^2+Y^2+Z^2)+\frac{K^2}{n}(f-c)(-2fY+nZ)$$
$$+\frac{K^4}{4n^2}\{n^2+bn+2f(f-c)\}=0,$$

where, in the second equation, the terms in Y, Z present themselves in the combination $-2fY+nZ$ with the constant coefficients $-2f$ and n. Hence writing

$$\sqrt{n^2+4f^2}\,Y=\ \ nY'-2fZ',$$

$$\sqrt{n^2+4f^2}\,Z=2fY'+\ \ nZ',$$

and consequently $-2fY+nZ=\sqrt{n^2+4f^2}\,Z'$, and (after the transformation) removing the accents, the equations become

$$-\alpha X+\frac{1}{\sqrt{n^2+4f^2}}[\{n^2+2f(f-\gamma)\}Y-n(f+\gamma)Z]=0,$$

$$X^2+Y^2+Z^2+\frac{K^2(f-c)\sqrt{n^2+4f^2}}{n\{n^2-bn+2f(f-c)\}}Z+\frac{K^4\{n^2+bn+2f(f-c)\}}{n^2\{n^2-bn+2f(f-c)\}}=0,$$

viz. interchanging the parameters so as to have t in the first equation and θ in the second equation, these are of the form

$$aX+bY+cZ=0,$$

$$X^2+Y^2+Z^2-2m\phi Z=\theta,$$

which belong to the case PS, 3^0. Hence, in this general case, the Inversion is a surface PS, 3^0.

I have spoken above of the particular case $e=0$, $f=0$: here the equations of the two sets of spheres are

$$x^2+y^2+z^2-2by\ -2cz=0,$$

$$x^2+y^2+z^2-2\alpha x-2\gamma z=0,$$

which have the origin as a common point. Taking this as the centre of inversion, or writing

$$x=\frac{K^2X}{\Omega},\quad y=\frac{K^2Y}{\Omega},\quad z=\frac{K^2Z}{\Omega},\ \text{where } \Omega=X^2+Y^2+Z^2,$$

the transformed equations are

$$bY+cZ-\tfrac{1}{2}K^2=0,$$

$$\alpha X\quad +\gamma Z-\tfrac{1}{2}K^2=0,$$

or, interchanging X and Y, say

$$bX\quad +cZ-\tfrac{1}{2}K^2=0,$$

$$\alpha Y+\gamma Z-\tfrac{1}{2}K^2=0,$$

which are of the form

$$X\quad +tZ-P\quad =0,$$

$$Y+\theta Z-\Pi\quad =0,$$

belonging to a surface PP, 3^0. Hence, in this case, the Inversion is a surface PP, 3^0.

It thus appears that the surface SS, 4^0 has an Inversion which is either PS, 3^0, PS, 4^0 or PP, 3^0. The inversion has in some cases to be performed in regard to an imaginary centre of inversion.

It was previously shown that the surface SS, 3^0 had an Inversion PS, 3^0, and we thus arrive at the conclusion that a surface SS, with its two sets of curves of curvature each spherical, is in every case the Inversion of a surface PS with one set plane and the other spherical, or else of a surface PP with each set plane. Serret notices that the centre of inversion may be imaginary: this (he says) presents no difficulty, but he adds that it is easy to see that the centres of inversion may be taken to be real, provided that we join to the surfaces thus obtained all the parallel surfaces.

It seems to me that there is room for further investigation as to the surfaces SS: first, without employing the theory of inversion, it would be desirable to obtain the several forms by direct integration, as was done in regard to the surfaces PP and PS; secondly, starting from the several surfaces PP and PS considered as known forms, it would be desirable to obtain from these, by inversion in regard to an arbitrary centre, or with regard to a centre in any special position, the several forms of the surfaces SS. But I do not at present propose to consider either of these questions.

In conclusion, I remark that I have throughout assumed Serret's *negative* conclusions, viz. that the several cases, other than those considered in the present memoir, give only developable surfaces, or else surfaces having circles for one set of their curves of curvature. These being excluded from consideration, there remain

PP, Serret's two cases PP, 1^0, PP, 3^0;

PS, his three cases PS, 1^0, PS, 3^0, PS, 4^0;

SS, his two cases SS, 2^0 and SS, 4^0;

but PP, 1^0 is a particular case of, and so may be included in, PP, 3^0; and similarly PS, 1^0 is a particular case of, and may be included in, PS, 3^0; the cases considered thus are

PP, 3^0; PS, 3^0, PS, 4^0; SS, 2^0 and SS, 4^0.

It would however appear by what precedes that the case SS, 4^0 includes several cases which it is possible might properly be regarded as distinct; and the classification of the surfaces SS can hardly be considered satisfactory; it would seem that there should be at any rate 3 cases, viz. the surfaces which are the Inversions of PP, 3^0, PS, 3^0 and PS, 4^0 respectively.

I regard the present memoir as a development of the analytical theory of the surfaces PP, 3^0, PS, 3^0 and PS, 4^0.