881.

ON HERMITE'S *H*-PRODUCT THEOREM.

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I GIVE this name to a theorem relating to the product of an even number of Eta-functions, established by M. Hermite in his "Note sur le calcul différential et le calcul integral," forming an appendix to the sixth edition of Lacroix's *Differential and Integral Calculus*, and separately printed, 8vo. Paris, 1862. It is the theorem stated p. 65, in the form

$$\phi(x) = F(z^2) + \frac{dz}{dx} zF_1(z^2),$$

where

$$\phi(x) = \frac{AH(x-\alpha_1)H(x-\alpha_2)\ldots H(x-\alpha_{2n})}{\Theta^{2n}(x)},$$

where $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n} = 0$, and $z = \operatorname{sn} x$, $\operatorname{cn} x$ or $\operatorname{dn} x$ at pleasure; $F(z^2)$, $F_1(z^2)$ denote rational and integral functions of z^2 of the degrees n and $n-2$ respectively; A is a constant, which we may if we please so determine that in $F(z^2)$ the coefficient of the highest power z^{2n} shall be $= 1$.

If, for shortness, we write s, c, d for $\operatorname{sn} x$, $\operatorname{cn} x$, $\operatorname{dn} x$ respectively; and to fix the ideas, assume $z = \operatorname{sn} x$, $= s$, then the theorem is

$$\frac{AH(x-\alpha_1)H(x-\alpha_2)\ldots H(x-\alpha_{2n})}{\Theta^{2n}(x)} = F(s^2) + scdF_1(s^2);$$

viz. the theorem is that the product of the $2n$ H-functions ($\alpha_1 + \alpha_2 + \ldots + \alpha_{2n} = 0$ as above), divided by $\Theta^{2n}(x)$, is a function of the elliptic functions sn, cn, dn, of the form in question.

Hermite uses the theorem for the demonstration of Abel's theorem, as applied to the elliptic functions; or as I would rather express it, he uses the theorem for the determination of the sn, cn, and dn of $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n-1}$.

To show how this is, observe that $F(s^2)$, *quà* rational and integral function of s^2 of the degree n, with its first coefficient $=1$, contains n arbitrary coefficients; and $F_1(s^2)$, *quà* rational and integral function of s^2 of the degree $n-2$, contains $n-1$ arbitrary coefficients: hence $F(s^2)+scdF_1(s^2)$ contains $2n-1$ arbitrary coefficients; and considering $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$ as given, the function in question must vanish for each of the values $x=\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$; and we have therefore $2n-1$ equations for obtaining the $2n-1$ coefficients, which are thus completely determined: in particular, the constant term, say L, of $F_1(s^2)$ is a given function of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$, that is, of the sn, cn, and dn of these quantities; and the theorem shows that the function thus determined vanishes also for $x=\alpha_{2n}$, that is, $=-(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$.

Now writing $-x$ for x in the formula, and recollecting that H is an odd function, Θ an even function, we find

$$\frac{AH(x+\alpha_1)\,H(x+\alpha_2)\ldots H(x+\alpha_{2n})}{\Theta^{2n}(x)}=F(s^2)-scdF_1(s^2);$$

and multiplying together the two sides of these equations respectively,

$$A^2\frac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)}\ldots\frac{H(x-\alpha_{2n})\,H(x+\alpha_{2n})}{\Theta^2(x)}=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2,$$

where the right-hand side is a rational and integral function of s^2 of the degree $2n$, and the coefficient of the highest term s^{4n} is $=1$; in fact, this term arises only from the square of $F(s^2)$, which has its highest term $=s^{2n}$.

Now $\dfrac{H(x-\alpha_1)\,H(x+\alpha_1)}{\Theta^2(x)}$ is a mere constant multiple of $\operatorname{sn}^2x-\operatorname{sn}^2\alpha_1$, or say of $s^2-\operatorname{sn}^2\alpha_1$; (this well-known theorem is, in fact, the particular case $n=2$ of Hermite's theorem); and similarly for the other terms: we must clearly have A^2, multiplied by the product of the factors thus introduced, $=1$; and thus the theorem becomes

$$(s^2-\operatorname{sn}^2\alpha_1)(s^2-\operatorname{sn}^2\alpha_2)\ldots(s^2-\operatorname{sn}^2\alpha_{2n})=\{F(s^2)\}^2-s^2c^2d^2\{F_1(s^2)\}^2.$$

And putting herein $s=0$, and writing as before L for the constant term of $F(s^2)$, we have

$$\operatorname{sn}^2\alpha_1\operatorname{sn}^2\alpha_2\ldots\operatorname{sn}^2\alpha_{2n}=L^2,$$

or, the sign $\pm$ being properly determined, say

$$\operatorname{sn}\alpha_1\operatorname{sn}\alpha_2\ldots\operatorname{sn}\alpha_{2n}=\pm L,$$

where, by what precedes, L is a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$. Hence we have $\operatorname{sn}\alpha_{2n}$, that is, $-\operatorname{sn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$.

Similarly writing $z=\operatorname{cn}x, =c$, and $z=\operatorname{dn}x, =d$, we have $\operatorname{cn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ and $\operatorname{dn}(\alpha_1+\alpha_2+\ldots+\alpha_{2n-1})$ each of them as a given function of the sn, cn, and dn of $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$.

It is hardly necessary to remark that $F(z^2)+z\dfrac{dz}{dx}F_1(z^2)$ is a function of the same form, whether we have $z=s$, c or d; in fact, the functions F and F_1 are rational in s^2, c^2, or d^2, and we have $z\dfrac{dz}{dx}=scd$, $-scd$, and $-k^2scd$ for the three values respectively.

The number of terms $\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}$ has been odd, but by taking one of them $=0$, the formulæ give the values of the sn, cn, and dn for the sum of an even number of terms.

It has been seen that Hermite's H-product theorem gives, say Abel's theorem, in the form

$$\Pi\,(s^2 - \text{sn}^2\,\alpha) = \{F(s^2)\}^2 - s^2c^2d^2\,\{F_1(s^2)\}^2,$$

each side of this relation being the product of two factors, viz. for the left-hand side the factors are

$$A\Pi\,\frac{H(x-\alpha)}{\Theta(x)},\quad A\Pi\,\frac{H(x+\alpha)}{\Theta(x)},$$

and for the right-hand side they are the rational functions of s^2,

$$F(s^2) + scdF_1(s^2),\quad F(s^2) - scdF_1(s^2);$$

these factors are by Hermite's theorem equal each to each; viz. this is the relation in which Hermite's stands to Abel's theorem.

The H-product theorem is given as one out of a group of four theorems; the other three may be called the H-product, H_1-product and Θ_1-product, odd theorems respectively,

$$\{\Theta_1(x) = \Theta(x+K),\quad H_1(x) = H(x+K)\},$$

viz. these are

$$\begin{cases} \dfrac{A_1 H(x-\alpha_1)\,H(x-\alpha_2)\ldots H(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = sF(s^2) + \quad cd\,\phi\,(s^2), \\[2ex] \dfrac{A_2 H_1(x-\alpha_1)\,H_1(x-\alpha_2)\ldots H_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = cF(c^2) - \quad sd\,\phi\,(c^2), \\[2ex] \dfrac{A_3 \Theta_1(x-\alpha_1)\,\Theta_1(x-\alpha_2)\ldots \Theta_1(x-\alpha_{2n+1})}{\Theta^{2n+1}(x)} = dF(d^2) - k^2sc\,\phi\,(d^2), \end{cases}$$

where F, ϕ are rational and integral functions of the degrees n and $n-1$, having their proper values in the three equations respectively, and in each case

$$\alpha_1 + \alpha_2 + \ldots + \alpha_{2n+1} = 0.$$

It was seen above that, for $n = 1$, the H-product theorem became

$$\frac{AH(x-\alpha)\,H(x+\alpha)}{\Theta^2(x)} = \text{sn}\;x - \text{sn}^2\,\alpha,$$

which is the most simple case; for the odd theorems, the most simple case is $n = 0$, viz. we then have

$$\frac{A_1 H(x)}{\Theta(x)} = \text{sn}\,x,\quad \frac{A_2 H_1(x)}{\Theta(x)} = \text{cn}\,x,\quad \frac{A_3 \Theta_1(x)}{\Theta(x)} = \text{dn}\,x;$$

to complete the formulæ observe that the values of the constants are

$$A = \frac{2k'K}{k\pi\Theta^2(\alpha)},\quad A_1 = \frac{1}{\sqrt{(k)}},\quad A_2 = \sqrt{\left(\frac{k'}{k}\right)},\quad A_3 = \sqrt{(k')}.$$

The three theorems may be used, in like manner with the H-product theorem, to give the values of the sn, cn, and dn respectively of the sum $\alpha_1 + \alpha_2 + \ldots + \alpha_{2n}$.