# 877.

## NOTE ON THE TWO RELATIONS CONNECTING THE DISTANCES OF FOUR POINTS ON A CIRCLE.

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CONSIDER a quadrilateral $BACD$ inscribed in a circle; and let the sides $BA$, $AC$, $CD$, $DB$ and diagonals $BC$ and $AD$ be $=c, b, h, g, a, -f$ respectively; $f$ is for convenience taken negative, so that the equation connecting the sides and diagonals may be

$$\Delta, = af + bg + ch, = 0.$$

We have between the sides and diagonals another relation

$$V, = abc + agh + bhf + cfg, = 0,$$

as is easily proved geometrically; in fact, recollecting that the opposite angles are supplementary to each other, the double area of the quadrilateral is $=(bc+gh)\sin A$, and it is also $=(bh+cg)\sin B$; that is, we have

$$(bc+gh)\sin A - (bh+cg)\sin B = 0.$$

But from the triangles $BAD$ and $BAC$, in which the angles $D$, $C$ are equal to each other, we have

$$\frac{c}{\sin D} = -\frac{f}{\sin B}, \quad \frac{c}{\sin C} = \frac{a}{\sin A};$$

that is,

$$f\sin A + a\sin B = 0;$$

and thence the required relation

$$a(bc+gh) + f(bh+cg) = 0.$$

The distances of the four points on the circle are thus connected by the two equations $\Delta = 0$, $V = 0$. Considering $a, b, c, f, g, h$ as the distances from each other of any four points in the plane, we have between them the relation

$$\begin{aligned}\Omega, = \;& a^2f^2(-a^2-f^2+b^2+g^2+c^2+h^2)\\ &+b^2g^2(\;\;a^2+f^2-b^2-g^2+c^2+h^2)\\ &+c^2h^2(\;\;a^2+f^2+b^2+g^2-c^2-h^2)\\ &-a^2b^2c^2-a^2g^2h^2-b^2h^2f^2-c^2f^2g^2, = 0;\end{aligned}$$

and it is clear that this equation should be a consequence of the equations $\Delta = 0$, $V = 0$. To verify this, forming the sum $\Omega + V^2$, we have

$$\begin{aligned}\Omega + V^2 = \;& (a^2+f^2)(-a^2f^2+b^2g^2+c^2h^2+2bgch)\\ &+(b^2+g^2)(-b^2g^2+c^2h^2+a^2f^2+2chaf)\\ &+(c^2+h^2)(-c^2h^2+a^2f^2+b^2g^2+2afbg);\end{aligned}$$

viz. this is

$$\begin{aligned}= \;& (a^2+f^2)\{-a^2f^2+(\Delta-af)^2\}\\ &+(b^2+g^2)\{-b^2g^2+(\Delta-bg)^2\}\\ &+(c^2+h^2)\{-c^2h^2+(\Delta-ch)^2\};\end{aligned}$$

or, since

$$-a^2f^2+(\Delta-af)^2 = \Delta(\Delta-2af) = \Delta(-af+bg+ch), \text{ \&c.},$$

this is

$$\begin{aligned}\Omega + V^2 = \Delta\,[\;& (a^2+f^2)(-af+bg+ch)\\ &+(b^2+g^2)(\;\;af-bg+ch)\\ &+(c^2+h^2)(\;\;af+bg-ch)],\end{aligned}$$

which proves the theorem.

It may be remarked that the equation $V = 0$ may be written

$$a(bc+gh)+f(bh+cg)=0;$$

viz. multiplying by $a$, and for $af$ writing its value, $=-(bg+ch)$ from the equation $\Delta = 0$, this gives

$$-a^2(bc+gh)+(bg+ch)(bh+cg)=0,$$

that is,

$$bc(g^2+h^2-a^2)+gh(b^2+c^2-a^2)=0,$$

which expresses that the angles $A$, $D$ are supplementary to each other; and, similarly, by the elimination of any other of the six quantities from the equations $\Delta = 0$, $V = 0$, we have five other like equations.