875.

ON THE SYSTEM OF THREE CIRCLES WHICH CUT EACH OTHER AT GIVEN ANGLES AND HAVE THEIR CENTRES IN A LINE.

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IN the system considered in the paper "System of equations for three circles which cut each other at given angles," *Messenger*, t. XVII. pp. 18—21, [873], we may consider the particular case where the centres of the circles are in a line. The condition in order that this may be so is obviously

$$\begin{vmatrix} \sin(A-\alpha)\cos F', & \sin(A-\alpha)\sin F', & \sin\alpha \\ \sin(B-\beta)\cos G', & \sin(B-\beta)\sin G', & \sin\beta \\ \sin(C-\gamma)\cos H', & \sin(C-\gamma)\sin H', & \sin\gamma \end{vmatrix} = 0,$$

that is,

$$\sin(B-\beta)\sin(C-\gamma)\sin\alpha\sin(G'-H') + \ldots = 0\,;$$

or since $\sin(G'-H')$, $\sin(H'-F')$, $\sin(F'-G')$ are $=\sin A$, $\sin B$, $\sin C$ respectively, this is

$$\sin(B-\beta)\sin(C-\gamma)\sin A\sin\alpha + \ldots = 0,$$

viz. this is

$$\frac{\sin A\,\sin\alpha}{\sin(A-\alpha)} + \frac{\sin B\sin\beta}{\sin(B-\beta)} + \frac{\sin C\,\sin\gamma}{\sin(C-\gamma)} = 0,$$

or, as this may also be written,

$$\frac{1}{\cot A - \cot\alpha} + \frac{1}{\cot B - \cot\beta} + \frac{1}{\cot C - \cot\gamma} = 0.$$

But assuming this equation to be satisfied, it does not appear that there is any simple expression for the equation of the line through the three centres; nor would it be easy to transform the equations so as to have this line for one of the axes.

The case in question (which is a very important one from its connexion with Poincaré's theory of the Fuchsian functions) may be considered independently.

Taking the line of centres for the axis of x, and writing α, β, γ for the abscissæ of the centres, and P, Q, R for the radii, then the equations of the circles are

$$(x-\alpha)^2+y^2=P^2,$$
$$(x-\beta)^2+y^2=Q^2,$$
$$(x-\gamma)^2+y^2=R^2;$$

and then, if the pairs of circles cut at the angles A, B, C respectively, we have

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$
$$R^2+2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2+2PQ\cos C+Q^2=(\alpha-\beta)^2,$$

which are the equations connecting α, β, γ, P, Q, R.

It is to be remarked in regard hereto that, if A, B, C are used to denote the interior angles of the curvilinear triangle ABC, then the angles $\gamma A\beta$, $\alpha B\gamma$, $\beta C\alpha$ are $=\pi-A$, B, C respectively; whence, if P, Q, R were used to denote the three radii taken positively, the first equation would be

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$

as above; but the other two equations would be

$$R^2-2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2-2PQ\cos C+Q^2=(\alpha-\beta)^2;$$

hence, in order that the equations may be as above, it is necessary that P denote the radius of the circle, centre α, taken *negatively*; and it in fact appears that, in a limiting case afterwards considered, the value of P comes out negative. Similarly as regards the curvilinear triangle $AB'C'$; here A, $B\,(=B')$ and $C\,(=C')$ are the interior angles of the triangle; and the radius of the circle, centre α', must be regarded as negative.

Considering A, B, C as given, we have an equation between the radii P, Q, R. In fact, this is at once obtained in the irrational form $\sqrt{(X)}+\sqrt{(Y)}+\sqrt{(Z)}=0$; and proceeding to rationalise this, we obtain

$$-2\sqrt{(YZ)}=Y+Z-X,$$

that is,

$$-\sqrt{\{(P^2+2PR\cos B+R^2)(P^2+2PQ\cos C+Q^2)\}}=P^2+P(Q\cos C+R\cos B)-QR\cos A.$$

Hence, squaring and reducing, we find without difficulty

$$\begin{aligned}0=\;&Q^2R^2\sin^2A+R^2P^2\sin^2B+P^2Q^2\sin^2C\\&+2P^2QR(\cos A+\cos B\cos C)+2PQ^2R(\cos B+\cos C\cos A)\\&+2PQR^2(\cos C+\cos A\cos B),\end{aligned}$$

or putting herein $P, Q, R = \frac{\sin A}{\xi}, \frac{\sin B}{\eta}, \frac{\sin C}{\zeta}$, this is

$$\left(1, 1, 1, \frac{\cos A + \cos B \cos C}{\sin B \sin C}, \frac{\cos B + \cos C \cos A}{\sin C \sin A}, \frac{\cos C + \cos A \cos B}{\sin A \sin B}\right)(\xi, \eta, \zeta)^2 = 0;$$

and it may be remarked that, in this quadric form, the three coefficients are each less than 1, or each greater than 1, according as $A + B + C > \pi$, or $A + B + C < \pi$.

Suppose 1°, $A + B + C > \pi$; the coefficients are here $= \cos\lambda$, $\cos\mu$, $\cos\nu$, the form is

$$(1, 1, 1, \cos\lambda, \cos\mu, \cos\nu)(\xi, \eta, \zeta)^2,$$

that is,

$$(\xi + \eta\cos\nu + \zeta\cos\mu)^2 + (\eta^2\sin^2\nu + 2\eta\zeta\cos\lambda + \zeta^2\sin^2\mu),$$

namely, this is

$$(\xi + \eta\cos\nu + \zeta\cos\mu)^2 + \left\{\eta\sin\nu + \zeta\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu}\right\}^2 + \zeta^2\left\{\sin^2\mu - \left(\frac{\cos\lambda - \cos\mu\cos\nu}{\sin\nu}\right)^2\right\};$$

where the last term is

$$= \frac{\zeta^2}{\sin^2\nu}\{\sin^2\mu\sin^2\nu - (\cos\lambda\cos\mu\cos\nu)^2\};$$

the coefficient in { } is

$$= 1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu,$$

namely, substituting for $\cos\lambda$, $\cos\mu$, $\cos\nu$ their values, this is

$$= \frac{1}{\sin^2 A\sin^2 B\sin^2 C}(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)^2.$$

It thus appears that the form is the sum of three squares, and is thus constantly positive: it therefore only vanishes for imaginary values of the radii; or the case does not arise for any real figure.

Hence, 2°, if the figure be real, $A + B + C < \pi$, that is, the sum of the angles of the curvilinear triangle is less than two right angles: the radii are connected as above by the equation

$$\left(1, 1, 1, \frac{\cos A + \cos B\cos C}{\sin B\sin C}, \frac{\cos B + \cos C\cos A}{\sin C\sin A}, \frac{\cos C + \cos A\cos B}{\sin A\sin B}\right)\left(\frac{\sin A}{P}, \frac{\sin B}{Q}, \frac{\sin C}{R}\right)^2 = 0,$$

in which form the three coefficients are each greater than 1. Restoring therein ξ, η, ζ, and regarding these as rectangular coordinates, the equation represents a real cone which might be constructed without difficulty; and then taking ξ, η, ζ as the coordinates of any particular point on the conical surface, we have

$$P, Q, R = \frac{\sin A}{\xi}, \frac{\sin B}{\eta}, \frac{\sin C}{\zeta}.$$

Obviously, points on the same generating line of the cone give values of P, Q, R which differ in their absolute magnitudes only, their ratios being the same: the original equations, in fact, remain unaltered when P, Q, R, α, β, γ are each affected with any common factor.

Supposing P, Q, R taken so as to satisfy the equation in question, then taking the radicals

$$\sqrt{(Q^2 + 2QR\cos A + R^2)},\quad \sqrt{(R^2 + 2RP\cos B + P^2)},\quad \sqrt{(P^2 + 2PQ\cos C + Q^2)},$$

with the proper signs, we have a sum $=0$, and these give the values of $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$, respectively; and the construction of the figure would be thus completed.

I look at the question from a different point of view; taking Q, R, $\beta-\gamma$ such as to satisfy the first equation

$$Q^2 + 2QR\cos A + R^2 = (\beta-\gamma)^2,$$

that is, starting from the two circles $(x-\beta)^2 + y^2 = Q^2$, $(x-\gamma)^2 + y^2 = R^2$ which cut each other at a given angle A: then the problem is to find a circle $(x-\alpha)^2 + y^2 = P^2$, cutting these at given angles C, B respectively. To determine the coordinate of the centre α, and the radius P, we have the remaining two equations

$$R^2 + 2RP\cos B + P^2 = (\gamma-\alpha)^2,$$
$$P^2 + 2PQ\cos C + Q^2 = (\alpha-\beta)^2,$$

namely, considering α, P as the coordinates of a point (in reference to the foregoing origin and axes), and for greater clearness writing $\alpha = \mathrm{x}$, $P = \mathrm{y}$, we have

$$\mathrm{y}^2 + 2\mathrm{y}R\cos B + R^2 - (\mathrm{x}-\gamma)^2 = 0,$$
$$\mathrm{y}^2 + 2\mathrm{y}Q\cos C + Q^2 - (\mathrm{x}-\beta)^2 = 0,$$

or, as these may be written,

$$(\mathrm{y} + R\cos B)^2 - (\mathrm{x}-\gamma)^2 = -R^2\sin^2 B,$$
$$(\mathrm{y} + Q\cos C)^2 - (\mathrm{x}-\beta)^2 = -Q^2\sin^2 C,$$

namely, the first of these equations denotes a rectangular hyperbola, coordinates of centre $(\mathrm{x}=\gamma,\ \mathrm{y} = -R\cos B)$, transverse semi-axes $= R\sin B$; and the second of them a rectangular hyperbola, coordinates of centre $(\mathrm{x}=\beta,\ \mathrm{y} = -Q\cos C)$, transverse semi-axes $= Q\sin C$: as similar and similarly situate hyperbolas, these intersect in two points only; namely, the points are the intersections of either of them with the common chord

$$2\mathrm{y}(R\cos B - Q\cos C) + 2(\gamma-\beta)\{\mathrm{x} - \tfrac{1}{2}(\gamma+\beta)\} + R^2 - Q^2 = 0.$$

It is possible to construct a circle through the two points of intersection, and so to obtain these points as the intersections of a line and circle; but the construction by the two rectangular hyperbolas is practically by no means an inconvenient one. I remark in passing that, for a rectangular hyperbola, the radius of curvature at the vertex is equal to the transverse semi-axis, and thus by drawing a small circular arc and by means of the asymptotes, we lay down a rectangular hyperbola graphically, without difficulty and with a fair amount of accuracy.

But the analytical solution may be carried somewhat further: we may without loss of generality write $\gamma = -\beta$, for this comes only to taking the origin midway between the centres of the circles β and γ: doing this, and for greater simplicity writing also for the moment $Q\cos C = M$, $R\cos B = N$, the equations become

$$y^2 + 2yN + R^2 - (x+\beta)^2 = 0,$$
$$y^2 + 2yM + Q^2 - (x-\beta)^2 = 0,$$

where x is now the abscissa of the centre of the circle α (measured from the last-mentioned midway point) and y is the radius of this circle. We deduce

$$2(N-M)\,y + R^2 - Q^2 - 4\beta x = 0,$$

or say

$$4\beta(x-\beta) = 2(N-M)\,y + R^2 - Q^2 + 4\beta^2,$$

and thence, from the first equation multiplied by $16\beta^2$, we have

$$16\beta^2(y^2 + 2yN + R^2) - \{2(N-M)\,y + R^2 - Q^2 + 4\beta^2\}^2 = 0,$$

that is,

$$\begin{aligned} &4y^2\{4\beta^2 - (N-M)^2\} \\ &+ 4y\,\{4\beta^2(N+M) - (N-M)(R^2-Q^2)\} \\ &+ \quad \{8\beta^2(R^2+Q^2) - 16\beta^4 - (R^2-Q^2)^2\} = 0, \end{aligned}$$

say this is $4y^2\mathfrak{A} + 4y\mathfrak{B} + \mathfrak{C} = 0$.

This gives

$$(2\mathfrak{A}y + \mathfrak{B})^2 = \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C},$$

and we find without difficulty, restoring for M, N their values $Q\cos C$ and $R\cos B$,

$$\begin{aligned} \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} = 4\beta^2\,\{&16\beta^4 - 8\beta^2(Q^2 + R^2 - 2QR\cos B\cos C) \\ &+ (P^4 - 4P^3Q\cos B\cos C + 2P^2Q^2(2\cos^2 B + 2\cos^2 C - 1) - 4PQ^3\cos B\cos C + Q^4)\}, \end{aligned}$$

which is

$$= 4\beta^2\,\{[4\beta^2 - (Q^2 - 2QR\cos B\cos C + R^2)]^2 - 4Q^2R^2\sin^2 B\sin^2 C\}.$$

But we have

$$Q^2 + 2QR\cos A + R^2 = 4\beta^2,$$

and this equation thus becomes

$$\begin{aligned} \mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} &= 16\beta^2Q^2R^2\,\{(\cos A + \cos B\cos C)^2 - \sin^2 B\sin^2 C\} \\ &= 16\beta^2Q^2R^2\,(-1 + \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C). \end{aligned}$$

We have therefore

$$\begin{aligned} 2\,\{4\beta^2 - (R\cos B - Q\cos C)^2\}\,y + \{4\beta^2(R\cos B + Q\cos C) - (R\cos B - Q\cos C)(R^2 - Q^2)\} \\ = \pm\, 4\beta QR\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}}, \end{aligned}$$

$$4(R\cos B - Q\cos C)\,y + R^2 - Q^2 = 4\beta x,$$

or, completing the reduction by the substitution of the value of $4\beta^2$, this is

$$\begin{aligned}\mathrm{y}\,\{(Q^2\sin^2 C + R^2\sin^2 B) + 2QR\,(\cos A + \cos B\cos C)\}\\ + QR\,\{Q\,(\cos B + \cos C\cos A) + R\,(\cos C + \cos A\cos B)\}\\ = \pm\, 4\beta QR\,\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}},\end{aligned}$$

viz. we have thus two values of the radius $\mathrm{y}\,(=P)$; and to each of these there corresponds a single value of the abscissa x, given by

$$4\beta\mathrm{x} = R^2 - Q^2 + 2\,(R\cos B - Q\cos C)\,\mathrm{y}.$$

The two values become equal, if $A + B \pm C = \pi$; in this case the three circles meet in a pair of points $(\mathrm{x}_1, \mathrm{y}_1)$, $(\mathrm{x}_1, -\mathrm{y}_1)$. In fact, writing $A + B + C = \pi$, and thence

$$\cos A = -\cos(B + C), \ = -\cos B\cos C + \sin B\sin C, \ \&c.,$$

we find

$$\begin{aligned}\{Q^2\sin^2 C + 2QR\,(\cos A + \cos B\cos C) + R^2\sin^2 B\}\,\mathrm{y}\\ + QR\,\{Q\,(\cos B + \cos C\cos A) + R\,(\cos C + \cos A\cos B)\} = 0,\end{aligned}$$

that is,

$$(Q\sin C + R\sin B)^2\,\mathrm{y} + QR\,(Q\sin C + R\sin B)\sin A = 0,$$

or, throwing out the factor $Q\sin C + R\sin B$, this is

$$(Q\sin C + R\sin B)\,\mathrm{y} + QR\sin A = 0,$$

and we then have

$$\begin{aligned}4\beta\mathrm{x} &= R^2 - Q^2 - 2\,(R\cos B - Q\cos C)\,\frac{QR\sin A}{R\sin B + Q\sin C}\\ &= \frac{1}{R\sin B + Q\sin C}\,\{(R\sin B + Q\sin C)(R^2 - Q^2) - 2\,(R\cos B - Q\cos C)\,QR\sin A\}.\end{aligned}$$

The term in { } is here

$$\begin{aligned}&R^3\,(\ \sin B)\\ &+ R^2Q\,(\ \sin C - 2\sin A\cos B)\\ &+ RQ^2\,(-\sin B + 2\sin A\cos C)\\ &+ \ Q^3\,(-\sin C),\end{aligned}$$

which is

$$\begin{aligned}= \ &R^3\,(\ \sin B)\\ &+ R^2Q\,(-\sin C + 2\sin B\cos A)\\ &+ RQ^2\,(\ \sin B - 2\sin C\cos A)\\ &+ \ Q^3\,(-\sin C)\\ &= (R^2 + Q^2 + 2RQ\cos A)(R\sin B - Q\sin C),\\ &= 4\beta^2\,(R\sin B - Q\sin C),\end{aligned}$$

or, finally

$$\mathrm{y} = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$\mathrm{x} = \frac{\beta\,(R\sin B - Q\sin C)}{R\sin B + Q\sin C}.$$

In these equations, y, x should be replaced by P, α respectively; and in obtaining them, it was assumed that $\gamma = -\beta$; restoring the general values of β, γ, the equations become

$$P = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$\alpha - \tfrac{1}{2}(\beta+\gamma) = \frac{\tfrac{1}{2}(\beta-\gamma)(R\sin B - Q\sin C)}{R\sin B + Q\sin C},$$

viz. this last equation becomes

$$\alpha = \frac{\beta R\sin B + \gamma Q\sin C}{R\sin B + Q\sin C},$$

or, say

$$\alpha(R\sin B + Q\sin C) - \beta R\sin B - \gamma Q\sin C = 0,$$

which by means of the first equation becomes

$$\alpha\frac{QR}{P}\sin A + \beta R\sin B + \gamma Q\sin C = 0.$$

It thus appears that the two equations are

$$\frac{\sin A}{P} + \frac{\sin B}{Q} + \frac{\sin C}{R} = 0,$$

$$\frac{\alpha\sin A}{P} + \frac{\beta\sin B}{Q} + \frac{\gamma\sin C}{R} = 0,$$

viz. these equations, wherein $A + B + C = \pi$, belong to the case where the three circles intersect in the same pair of points; hence, if the coordinates x, y refer to the points of intersection of the three circles, we have simultaneously the equations of the three circles, and the three equations which determine the angles at which they intersect, viz. we have the six equations

$$(x-\alpha)^2 + y^2 = P^2, \quad Q^2 + R^2 + 2QR\cos A = (\beta-\gamma)^2,$$
$$(x-\beta)^2 + y^2 = Q^2, \quad R^2 + P^2 + 2RP\cos B = (\gamma-\alpha)^2,$$
$$(x-\gamma)^2 + y^2 = R^2, \quad P^2 + Q^2 + 2PQ\cos C = (\alpha-\beta)^2,$$

viz. from these six equations, with the condition $A + B + C = \pi$, it must be possible to deduce the last-mentioned pair of equations.

In the general case, where $A + B + C < \pi$, and the three circles do not meet in a point, then taking the circles $(x-\beta)^2 + y^2 = Q^2$, $(x-\gamma)^2 + y^2 = R^2$ to be circles cutting each other at the angle A, or, what is the same thing, the values Q, R, β, γ to be such as to satisfy the relation

$$Q^2 + R^2 + 2QR\cos A = (\beta-\gamma)^2;$$

the two equations for the determination of the abscissa of the centre α, and the radius P of the remaining circle give, by what precedes,

$$\begin{aligned} &2\{(\beta-\gamma)^2 - (R\cos B - Q\cos C)^2\}P \\ &\quad + \{(\beta-\gamma)^2(R\cos B + Q\cos C) - (R\cos B - Q\cos C)(R^2 - Q^2)\} \\ &\quad = \pm 2(\beta-\gamma)QR\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}}, \\ &4(R\cos B - Q\cos C)P + (R^2 - Q^2) = (\beta-\gamma)(2\alpha - \beta - \gamma), \end{aligned}$$

viz. we have thus the two circles $(x-\alpha)^2 + y^2 = P^2$, each of them cutting the circles $(x-\beta)^2 + y^2 = Q^2$, and $(x-\gamma)^2 + y^2 = R^2$ at the angles C, B respectively.