874.

NOTE ON THE LEGENDRIAN COEFFICIENTS OF THE SECOND KIND.

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AS regards the integration of the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

(n a positive integer), it seems to me that sufficient prominence is not given to the solution

$$y=\tfrac{1}{2}P_n\log\frac{x+1}{x-1}-Z_n\,(=Q_n),$$

where P_n is the Legendrian integral of the first kind, a rational and integral function of x of the degree n, and Z_n is a rational and integral function of the degree $n-1$; viz. we have here a solution containing no transcendental function other than the logarithm, and which should thus be adopted as a second particular integral in preference to the form $y=Q_n$, in which we have the infinite series Q_n which is an unknown transcendental function.

Moreover, the expression usually given for Z_n, viz.

$$Z_n=\frac{2n-1}{1\,.\,n}P_{n-1}+\frac{2n-5}{3(n-1)}P_{n-3}+\frac{2n-9}{5(n-2)}P_{n-5}+\ldots\text{(to term in }P_1\text{ or }P_0\text{)},$$

is a very simple and elegant one; but the more natural definition (and that by which Z_n is most readily calculated) is that Z_n is the integral part of $\frac{1}{2}P_n\log\frac{x+1}{x-1}$, when the logarithm is expanded in descending powers of x, viz. it is the integral part of

$$P_n\left(\frac{1}{x}+\frac{1}{3}\,\frac{1}{x^3}+\frac{1}{5}\,\frac{1}{x^5}+\ldots\right),$$

whence also Q_n is the portion containing negative powers only of this same series.

The expressions for P_0, $P_1, \ldots, P_{10}$ are given in Ferrers' *Elementary Treatise on Spherical Harmonics, &c.*, London, 1877, pp. 23—25. Reproducing these, and joining to them the values of Z_0, $Z_1, \ldots, Z_{10}$ we have as follows: read $P_2 = \frac{3}{2}x^2 - \frac{1}{2}$, and so in other cases.

$$P_0 = \quad 1 \ ,$$
$$P_1 = x \quad 1 \ ,$$
$$P_2 = (x^2, \ 1) \ \tfrac{3}{2} - \tfrac{1}{2} \ ,$$
$$P_3 = (x^3, \ x) \ \tfrac{5}{2} - \tfrac{3}{2} \ ,$$
$$P_4 = (x^4 \ldots 1) \ \tfrac{35}{8} - \tfrac{15}{4} + \tfrac{3}{8} \ ,$$
$$P_5 = (x^5 \ldots x) \ \tfrac{63}{8} - \tfrac{35}{4} + \tfrac{15}{8} \ ,$$
$$P_6 = (x^6 \ldots 1) \ \tfrac{231}{16} - \tfrac{315}{16} + \tfrac{105}{16} - \tfrac{5}{16} \ ,$$
$$P_7 = (x^7 \ldots x) \ \tfrac{429}{16} - \tfrac{693}{16} + \tfrac{315}{16} - \tfrac{35}{16} \ ,$$
$$P_8 = (x^8 \ldots 1) \ \tfrac{6435}{128} - \tfrac{3003}{32} + \tfrac{3465}{64} - \tfrac{315}{32} + \tfrac{35}{128} \ ,$$
$$P_9 = (x^9 \ldots x) \ \tfrac{12155}{128} - \tfrac{6435}{32} + \tfrac{9009}{64} - \tfrac{1155}{32} + \tfrac{315}{128} \ ,$$
$$P_{10} = (x^{10} \ldots 1) \ \tfrac{46189}{256} - \tfrac{109395}{256} + \tfrac{45045}{128} - \tfrac{15015}{128} + \tfrac{3465}{256} - \tfrac{63}{256} .$$
$$Z_0 = \quad 0 \ ,$$
$$Z_1 = \quad 1 \ ,$$
$$Z_2 = x \quad \tfrac{3}{2} \ ,$$
$$Z_3 = (x^2, \ 1) \ \tfrac{5}{2} - \tfrac{2}{3} \ ,$$
$$Z_4 = (x^3, \ x) \ \tfrac{35}{8} - \tfrac{55}{24} \ ,$$
$$Z_5 = (x^4 \ldots 1) \ \tfrac{63}{8} - \tfrac{49}{8} + \tfrac{8}{15} \ ,$$
$$Z_6 = (x^5 \ldots x) \ \tfrac{231}{16} - \tfrac{119}{8} + \tfrac{231}{80} \ ,$$
$$Z_7 = (x^6 \ldots 1) \ \tfrac{429}{16} - \tfrac{275}{8} + \tfrac{849}{80} - \tfrac{16}{35} \ ,$$
$$Z_8 = (x^7 \ldots x) \ \tfrac{6435}{128} - \tfrac{9867}{128} + \tfrac{4213}{128} - \tfrac{11659}{4480} \ ,$$
$$Z_9 = (x^8 \ldots 1) \ \tfrac{12155}{128} - \tfrac{65065}{384} + \tfrac{11869}{128} - \tfrac{14179}{896} + \tfrac{128}{315} \ ,$$
$$Z_{10} = (x^9 \ldots x) \ \tfrac{46189}{256} - \tfrac{281996}{768} + \tfrac{157157}{640} - \tfrac{26741}{448} + \tfrac{61567}{16128} .$$

I notice that the numerical values of P_1, $P_2, \ldots, P_7$, for $x = 0{\cdot}00$, $0{\cdot}01$, …, $1{\cdot}00$ are given (*Report of the British Association for* 1879, "Report on Mathematical Tables"); as the functions contain only powers of 2 in their denominators, the decimal values terminate, and the complete values are given. The functions Z have not been tabulated; the denominators contain other prime factors, and the decimal values would not terminate.

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