871.

A CASE OF COMPLEX MULTIPLICATION WITH IMAGINARY MODULUS ARISING OUT OF THE CUBIC TRANSFORMATION IN ELLIPTIC FUNCTIONS.

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THE case in question is referred to in my "Note on the Theory of Elliptic Integrals," *Math. Ann.*, t. XII. (1877), pp. 143—146, [657]; but I here work it out directly.

In the cubic transformation, the modular equation is

$$u^4 - v^4 + 2uv\,(1 - u^2v^2) = 0;$$

and we have

$$y = \frac{\left(1 + \frac{2u^3}{v}\right) x + \frac{u^6}{v^2}\,x^3}{1 + vu^2\,(v + 2u^3)\,x^2},$$

giving

$$\frac{dy}{\sqrt{1 - y^2 \,.\, 1 - v^8y^2}} = \frac{\left(1 + \frac{2u^3}{v}\right) dx}{\sqrt{1 - x^2 \,.\, 1 - u^8x^2}}.$$

We thus have a case of complex multiplication if $v^8 = u^8$, or say $v = \gamma u$, where $\gamma^8 = 1$, or γ denotes an eighth root of unity. Substituting in the modular equation, this becomes

$$u^4\,(1 - \gamma^4) + 2\gamma u^2\,(1 - \gamma^2 u^4) = 0,$$

or, throwing out the factor u^2 and reducing,

$$u^4 - \tfrac{1}{2}u^2\,(\gamma^5 - \gamma) - \gamma^6 = 0,$$

that is,

$$\frac{u^2}{\gamma} = \tfrac{1}{4}\,(\gamma^4 - 1 \pm \sqrt{\gamma^8 + 14\gamma^4 + 1}),$$

or, what is the same thing,

$$= \tfrac{1}{4}\{\gamma^4 - 1 \pm \sqrt{14\gamma^4 + 2}\}.$$

We have $\gamma^8 = 1$, that is, $\gamma^4 = \pm 1$. Considering first the case $\gamma^4 = 1$, here

$$\frac{u^2}{\gamma} = \pm 1,$$

and thence

$$1 + \frac{2u^3}{v} = 1 + \frac{2u^2}{\gamma}, \ = 1 \pm 2, \ = 3 \text{ or } -1;$$

moreover, $u^8 = v^8 = 1$. We have thus only the non-elliptic formulæ

$$\frac{dy}{1-y^2} = \frac{-dx}{1-x^2}, \text{ satisfied by } y = -x,$$

and

$$\frac{dy}{1-y^2} = \frac{3dx}{1-x^2}, \qquad \text{by } y = \frac{3x + x^3}{1+3x^2}.$$

If however, $\gamma^4 = -1$, then

$$\frac{u^2}{\gamma} = \tfrac{1}{4}(-2 \pm \sqrt{-12}),$$

viz. this is

$$\frac{u^2}{\gamma} = \tfrac{1}{2}(-1 \pm i\sqrt{3}) = \omega,$$

if ω be an imaginary cube root of unity $(\omega^2 + \omega + 1 = 0)$; hence

$$u^8 = (\gamma\omega)^4 = -\omega.$$

Moreover,

$$1 + \frac{2u^3}{v} = 1 + \frac{2u^2}{\gamma}, \ = 1 + 2\omega,$$

or say,

$$= \omega - \omega^2, \quad [= \sqrt{-3}, \text{ if } \omega = \tfrac{1}{2}(-1 + i\sqrt{3})];$$

and we thus have, as in the above-mentioned Note,

$$y = \frac{(\omega - \omega^2)\,x + \omega^2 x^3}{1 - \omega^2(\omega - \omega^2)\,x^2},$$

giving

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1+\omega y^2}} = \frac{(\omega - \omega^2)\,dx}{\sqrt{1-x^2 \,.\, 1+\omega x^2}};$$

or, what is the same thing, for the modulus $k^2 = -\omega$, we have

$$\text{sn}\,(\omega - \omega^2)\,\theta = \frac{(\omega - \omega^2)\,\text{sn}\,\theta + \omega^2\,\text{sn}^3\,\theta}{1 - \omega^2(\omega - \omega^2)\,\text{sn}^2\,\theta};$$

the values of $\text{cn}\,(\omega - \omega^2)\,\theta$ and $\text{dn}\,(\omega - \omega^2)\,\theta$ are thence found to be

$$\text{cn}\,(\omega - \omega^2)\,\theta = \frac{\text{cn}\,\theta\,(1 - \omega^2\,\text{sn}^2\,\theta)}{1 - \omega^2(\omega - \omega^2)\,\text{sn}^2\,\theta},$$

and

$$\text{dn}\,(\omega - \omega^2)\,\theta = \frac{\text{dn}\,\theta\,(1 + \omega^2\,\text{sn}^2\,\theta)}{1 - \omega^2(\omega - \omega^2)\,\text{sn}^2\,\theta};$$

which are the formulæ of transformation for the elliptic functions.