870.

ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS (SEQUEL).

[From the *American Journal of Mathematics*, vol. x. (1888), pp. 71—93.]

THE chief object of the present paper is the further development of the $\rho\alpha\beta$-theory in the case $n=7$. I recall that the forms are

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}}=\frac{\rho dx}{\sqrt{1-2\alpha x^2+x^4}},$$

where

$$y=\frac{x(\rho+A_2x^2+A_1x^4+x^6)}{1+A_1x^2+A_2x^4+\rho x^6}.$$

The paragraphs are numbered consecutively with those of the former paper "On the Transformation of Elliptic Functions," vol. IX., pp. 193—224, [869].

The Seventhic Transformation: the $\rho\alpha$-Equation. Art. Nos. 51 to 57.

51. The equation is given incorrectly Nos. 7 and 42; there was an error of sign in a term $512\alpha^3\rho$, which affected also the coefficient of $\alpha\rho$, and an error of sign in the absolute term 7. The correct form is

$$\rho^8-28\rho^6-112\alpha\rho^5-210\rho^4-224\alpha\rho^3+(-1484+1344\alpha^2)\rho^2+(464\alpha-512\alpha^3)\rho-7=0;$$

or, arranging in powers of α, this is

$$\begin{aligned}&\alpha^3\,.\,512\rho\\&+\alpha^2\,.-1344\rho^2\\&+\alpha\,.\,112\rho^5+224\rho^3-464\rho\\&-(\rho^8-28\rho^6-210\rho^4-1484\rho^2-7)=0.\end{aligned}$$

This may also be written in the forms

$$(\alpha-1)\{\alpha^2.512\rho+\alpha(-1344\rho^2+512\rho)+112\rho^5+224\rho^3-1344\rho^2+48\rho\}-(\rho+1)^7(\rho-7)=0,$$

and

$$(\alpha+1)\{\alpha^2.512\rho+\alpha(-1344\rho^2-512\rho)+112\rho^5+224\rho^3+1344\rho^2+48\rho\}-(\rho-1)^7(\rho+7)=0.$$

To simplify the $\rho\alpha$-equation, we assume $A=8\rho\alpha-7\rho^2$; then the $A\rho$-equation is

$$\begin{aligned}&A^3\\&+A\rho^2(14\rho^4-119\rho^2-58)\\&-\ \rho^2(\rho^8-126\rho^6+280\rho^4-1078\rho^2-7)=0;\end{aligned}$$

viz. this is a cubic equation wanting its second term, and so at once solvable by Cardan's formula: say the equation is

$$A^3+A\rho^2q_1-\rho^2r_1=0,$$

where

$$\begin{aligned}q_1&=14\rho^4-119\rho^2-58,\\r_1&=\ \rho^8-126\rho^6+280\rho^4-1078\rho^2-7.\end{aligned}$$

It is convenient to recall here that, writing $\sigma=-\dfrac{7}{\rho}$, and $B=8\sigma\beta-7\sigma^2$, we have between σ, β, B precisely the same equations as between ρ, α, A; $\rho=1$ gives $\sigma=-7$, and we have as corresponding values $\alpha=-1$, $A=-15$, $\beta=-1$, $B=-287$: these are very convenient for verification of the formulæ. Similarly, $\rho=-7$ gives $\sigma=1$, and then $\alpha=-1$, $A=-287$, $\beta=-1$, $B=-15$; but I have, in general, used the former values only.

52. We have

$$A=f+g,$$

where

$$\begin{aligned}3fg&=-\rho^2q_1,\\f^3+g^3&=\ \rho^2r_1,\end{aligned}$$

and thence

$$f^3-g^3=\rho^2\sqrt{r_1^2+\frac{4\rho^2q_1^3}{27}}.$$

We have identically

$$27(\rho^8-126\rho^6+280\rho^4-1078\rho^2-7)^2+4\rho^2(14\rho^4-119\rho^2-58)^3$$
$$=(\rho^6+75\rho^4-141\rho^2+1)^2(27\rho^4+122\rho^2+1323):$$

if $\rho=1$, this is $27.930^2+4(-163)^3=64^2.1472$; that is, $23352300-17322988=6029312$, which is right; but it is convenient to divide by 27, so as instead of $27\rho^4+122\rho^2+1323$ to have in the formulæ $\rho^4+\frac{122}{27}\rho^2+49$, or say

$$\rho^4+K\rho^2+49\ \ (K=\tfrac{122}{27}).$$

Hence writing

$$t_1 = \rho^6 + 75\rho^4 - 141\rho^2 + 1,$$
$$\delta = \rho^4 + K\rho^2 + 49,$$

we have

$$r_1^2 + \tfrac{4}{27}\rho^2 q_1 = t_1^2\delta,$$

and consequently

$$2f^3 = \rho^2(r_1 + t_1\sqrt{\delta}),$$
$$2g^3 = \rho^2(r_1 - t_1\sqrt{\delta}).$$

53. It was easy to foresee that the cube root of $r_1 \pm t_1\sqrt{\delta}$ would break up into the form $(U \pm \sqrt{\delta})\sqrt[3]{W \pm \sqrt{\delta}}$, and I was led to the actual expressions by the identities

$$20(14\rho^4 - 119\rho^2 - 58) = (19\rho^2 - 53)^2 - 3(27\rho^4 + 122\rho^2 + 1323);$$

that is,

$$20q_1 \qquad = (19\rho^2 - 53)^2 - 81\delta,$$

and

$$27(\rho^2 - 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = -500\rho^2,$$
$$27(\rho^2 + 7)^2 - (27\rho^4 + 122\rho^2 + 1323) = \quad 256\rho^2;$$

or, as these may be written,

$$(\rho^2 - 7)^2 - \delta = -\tfrac{500}{27}\rho^2, \quad (\rho^2 + 7)^2 - \delta = \tfrac{256}{27}\rho^2.$$

We, in fact, have further the two identities

$$\begin{aligned}1000(\rho^6 + 75\rho^4 - 141\rho^2 + 1) &= \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\} \\ &\quad + \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7),\end{aligned}$$

$$\begin{aligned}-1000(\rho^8 - 126\rho^6 + 280\rho^4 - 1078\rho^2 - 7) &= \{(19\rho^2 - 53)^3 + 243(19\rho^2 - 53)(\rho^4 + K\rho^2 + 49)\}(-\rho^2 + 7) \\ &\quad + \{27(19\rho^2 - 53)^2 \qquad + 729(\rho^4 + K\rho^2 + 49)\}(\rho^4 + K\rho^2 + 49),\end{aligned}$$

viz. writing

$$19\rho^2 - 53 = 9U, \quad -\rho^2 + 7 = W,$$

these equations become

$$\tfrac{1000}{729}t_1 = U^3 + 3U\delta + (3U^2 + \delta)W,$$
$$-\tfrac{1000}{729}r_1 = (U^3 + 3U\delta)W + (3U^2 + \delta)\delta,$$

and we have thus

$$-\tfrac{1000}{729}(r_1 - t_1\sqrt{\delta}) = (U + \sqrt{\delta})^3(W + \sqrt{\delta}),$$

and the like equation with $-\sqrt{\delta}$ in place of $\sqrt{\delta}$.

54. In part verification of the last-mentioned identities, observe that, in the first of them, putting $\rho = 1$, and comparing first the coefficients of ρ^6 and then the coefficients of ρ^0, we ought to have

$$1000 = 19^3 + 243 \,.\, 19 - (27 \,.\, 19^2 + 729), \; = 11476 - 10476,$$
$$1000 = (-53^3 - 243 \,.\, 53 \,.\, 49) + (27 \,.\, 53^2 + 729 \,.\, 49)\,7, \; = -779948 + 780948,$$

which are right; and similarly in the second equation, comparing first the coefficients of ρ^8 and next those of ρ^0, we have

$$-1000 = (19^3 + 243 . 19)(-1) + (27 . 19^2 + 729), = -11476 + 10476,$$

$$+7000 = (-53^3 - 243 . 53 . 49)(7) + (27 . 53^2 + 729 . 49) 49,$$

$$= -5459636 + 5466636,$$

which are right.

55. We have now $A = f + g$, where

$$f = -\tfrac{9}{10}(U - \sqrt{\delta}) \sqrt[3]{\tfrac{1}{2}\rho^2 (W - \sqrt{\delta})},$$

$$g = -\tfrac{9}{10}(U + \sqrt{\delta}) \sqrt[3]{\tfrac{1}{2}\rho^2 (W + \sqrt{\delta})}:$$

observe that, multiplying these two values, we have

$$fg = \tfrac{81}{100}(U^2 - \delta) \sqrt[3]{\tfrac{1}{4}\rho^4 . (W^2 - \delta)}, = \tfrac{81}{100}(U^2 - \delta) \sqrt[3]{\tfrac{1}{4}\rho^4 . \frac{-500}{27}\rho^2},$$

$$= \tfrac{81}{100}(U^2 - \delta)(-\tfrac{5}{3}\rho^2);$$

that is,

$$fg = -\tfrac{27}{20}\rho^2 (U^2 - \delta), = -\tfrac{27}{20}\rho^2 . \frac{20 q_1}{81} = -\tfrac{1}{3}\rho^2 q_1,$$

which is right. Or, finally, substituting for U, W, δ their values, we have, for the solution of the $A\rho$-equation, $A = f + g$, where

$$f = -\tfrac{9}{10}(19\rho^2 - 53 - \sqrt{\rho^4 + K\rho^2 + 49}) \sqrt[3]{\tfrac{1}{2}\rho^2 \{-\rho^2 + 7 - \sqrt{\rho^4 + K\rho^2 + 49}\}}, \quad (K = \tfrac{122}{27}),$$

$$g = -\tfrac{9}{10}(19\rho^2 - 53 + \sqrt{\rho^4 + K\rho^2 + 49}) \sqrt[3]{\tfrac{1}{2}\rho^2 \{-\rho^2 + 7 + \sqrt{\rho^4 + K\rho^2 + 49}\}}.$$

56. In the case $\rho = 1$, α has a value $= -1$, giving for A, $= 8\rho\alpha - 7\rho^2$, the value -15; and, in fact, here $\rho^2 = 1$, and the $A\rho$-equation becomes

$$A^3 - 163A + 930 = 0,$$

that is,

$$(A + 15)(A^2 - 15A + 62) = 0,$$

the roots thus being

$$A = -15, \quad A = \tfrac{1}{2}(15 \pm i\sqrt{23}).$$

To verify in this case the values given by the solution of the cubic equation, observe that, for $\rho^2 = 1$, we have $\delta = 50 + \frac{122}{27}$, $= \frac{1472}{27}$, and therefore $\sqrt{\delta} = \dfrac{8\sqrt{23}}{3\sqrt{3}}$, $= \dfrac{8\sqrt{69}}{9}$; also, $U = \dfrac{19\rho^2 - 53}{9}$, $= \dfrac{-34}{9}$, and $W = -\rho^2 + 7$, $= 6$. Hence $U + \sqrt{\delta} = \dfrac{-34 + 8\sqrt{69}}{9}$, and

$$\sqrt[3]{\tfrac{1}{2}\rho^2 (W + \sqrt{\delta})} = \sqrt[3]{3 + \frac{8\sqrt{69}}{9}}, = \frac{\sqrt[3]{81 + 12\sqrt{69}}}{3};$$

hence

$$g = -\tfrac{9}{10}\frac{2(-17 + 4\sqrt{69})}{9}\tfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}}, = \tfrac{1}{15}(17 - 4\sqrt{69})\sqrt[3]{81 + 12\sqrt{69}};$$

but the cube root is $=\frac{1}{2}(3+\sqrt{69})$, and we have $(17-4\sqrt{69})(3+\sqrt{69})=-225+5\sqrt{69}$, $=5(-45+\sqrt{69})$; that is, $g=\frac{1}{6}(-45+\sqrt{69})$. Similarly, $f=\frac{1}{6}(-45-\sqrt{69})$. We have thus the real root $f+g=-15$, and the imaginary roots

$$f\omega+g\omega^2 \text{ or } f\omega^2+g\omega, \ =-\tfrac{15}{2}(\omega+\omega^2)+\tfrac{1}{6}\sqrt{69}(\omega-\omega^2),$$

viz. the first term is $=\frac{15}{2}$ and the second is $\pm\frac{1}{6}\sqrt{69}\,.\,i\sqrt{3}$, $=\pm\frac{1}{2}i\sqrt{23}$; thus the roots are $\frac{1}{2}(15\pm i\sqrt{23})$, as they should be.

57. I found, by considerations arising out of the new theory Nos. 72 *et seq.*, that writing for shortness $m=i\sqrt{3}$, then, for $\rho=m-2$, the $\rho\alpha$-equation has a root $\alpha=m$; the corresponding values of $A_1\rho^2$ thus are $A=12m-31$, $\rho^2=-4m+1$, viz. substituting this value for ρ^2 in the $A\rho$-equation, there should be a root $A=12m-31$. The equation becomes

$$A^3+A(3704m-7653)+148306m+206162=0,$$

or, as this may be written,

$$(A-12m+31)\{A^2+A(12m-31)+2960m+4062\}=0,$$

and the roots thus are

$$A=\ \ 12m-31,$$
$$A=-6m+\tfrac{31}{2}\pm\tfrac{1}{2}\sqrt{-12584m-16777},$$

where the square root is not expressible as a rational function of m.

Expression of β as a Rational Function of α, ρ. Art. Nos. 58 to 66.

58. Writing $\sigma=-\dfrac{7}{\rho}$, we have β the same function of σ that ρ is of α; hence if $B=8\sigma\beta-7\sigma^2$, the $B\sigma$-equation is

$$\begin{aligned}&B^3\\&+B\sigma^2(14\sigma^4-119\sigma^2-58)\\&-\sigma^2(\sigma^8-126\sigma^6+280\sigma^4-1078\sigma^2-7)=0;\end{aligned}$$

and the expression for B in terms of σ is obtained from that of A by the mere change of ρ into σ. Say we have $B=f'+g'$, where

$$f'=-\tfrac{9}{10}(U'-\sqrt{\delta'})\sqrt[3]{\tfrac{1}{2}\sigma^2(W'-\sqrt{\delta'})},$$
$$g'=-\tfrac{9}{10}(U'+\sqrt{\delta'})\sqrt[3]{\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})};$$

then we have

$$\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})=\tfrac{1}{2}\frac{49}{\rho^2}\left(-\frac{49}{\rho^2}+7+\sqrt{\frac{2401}{\rho^4}+\frac{49K}{\rho^2}+49}\right)$$

$$=-\tfrac{1}{2}\,.\,\frac{343}{\rho^4}(-\rho^2+7-\sqrt{\rho^4+K\rho^2+49})$$

$$=-\frac{343}{\rho^6}\,.\,\tfrac{1}{2}\rho^2(W-\sqrt{\delta}),$$

or, say

$$\sqrt[3]{\tfrac{1}{2}\sigma^2(W'-\sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\tfrac{1}{2}\rho^2(W-\sqrt{\delta})};$$

and similarly

$$\sqrt[3]{\tfrac{1}{2}\sigma^2(W'+\sqrt{\delta'})} = -\frac{7}{\rho^2}\sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})}.$$

The cube roots which enter into the expression of B are thus identical with those in the expression of A, and it hence appears that B can be expressed rationally in terms of A, ρ; or, what is the same thing, β can be expressed rationally in terms of α, ρ.

59. The *à priori* reason is obvious: the $\rho\alpha$-equation is a cubic in α, but of the order 8 in ρ; hence, to a given value of α, there correspond 8 values of ρ. Similarly, the $\sigma\beta$-equation is a cubic in β, but it is of the order 8 in σ; or if for σ we substitute its value $= -\frac{7}{\rho}$, then we have a $\rho\beta$-equation which is a cubic in β, but it is of the order 8 in ρ. In the absence of any special relation between this $\rho\beta$-equation and the $\rho\alpha$-equation, there would correspond, to each of the 8 values of ρ, 3 values of β; that is, to a given value of α there would correspond $8 \times 3, = 24$ values of β. But, in fact, to a given value of α, there correspond only 8 values of β, and the two cubic equations are related to each other in such wise that this is so; viz. the relation between them is such that it is possible by means of them to express β as a rational function of ρ, α.

60. Returning to the investigation, we have

$$9U' = 19\sigma^2 - 53, = \frac{19.49}{\rho^2} - 53;$$

or, writing

$$63\bar{U} = 53\rho^2 - 931,$$

this is

$$U' = -\frac{7}{\rho^2}\bar{U}, \text{ whence } U' \pm \sqrt{\delta'} = -\frac{7}{\rho^2}(\bar{U} \mp \sqrt{\delta}).$$

Hence writing

$$\theta = \sqrt[3]{\tfrac{1}{2}\rho^2(W-\sqrt{\delta})}, \quad \phi = \sqrt[3]{\tfrac{1}{2}\rho^2(W+\sqrt{\delta})},$$

we have

$$f = -\tfrac{9}{10}(U-\sqrt{\delta})\,\theta, \quad f' = -\tfrac{9}{10}\frac{49}{\rho^4}(\bar{U}+\sqrt{\delta})\,\phi,$$

$$g = -\tfrac{9}{10}(U+\sqrt{\delta})\,\phi, \quad g' = -\tfrac{9}{10}\frac{49}{\rho^4}(\bar{U}-\sqrt{\delta})\,\theta,$$

so that, putting for shortness

$$L = -\tfrac{9}{10}(U-\sqrt{\delta}) \quad , \quad \bar{L} = -\tfrac{9}{10}\frac{49}{\rho^4}(\bar{U}-\sqrt{\delta}),$$

$$M = -\tfrac{9}{10}(U+\sqrt{\delta}) \quad , \quad \bar{M} = -\tfrac{9}{10}\frac{49}{\rho^4}(\bar{U}+\sqrt{\delta}),$$

we have

$$A = L\theta + M\phi, \quad B = \bar{L}\theta + \bar{M}\phi,$$

where θ^3, ϕ^3 and $\theta\phi$ are each of them free from any cube root; we have, in fact,

$$\theta\phi = \sqrt[3]{\tfrac{1}{4}\rho^4(W^2 - \delta)}, \quad = \sqrt[3]{\tfrac{1}{4}\rho^4 \cdot \frac{-500}{27}\rho^2}, \quad = -\tfrac{5}{3}\rho^2,$$

and it may be added that

$$3LM\theta\phi = -\rho^2 q_1, \text{ whence } LM = \tfrac{1}{5}q_1,$$

$$L^3\theta^3 + M^3\phi^3 = \quad \rho^2 r_1,$$

$$L^3\theta^3 - M^3\phi^3 = \quad \rho^2 t_1\sqrt{\delta};$$

these are, in fact, only the equations obtained by writing $L\theta$, $M\phi$ in place of f, g respectively.

61. In the case $\rho = 1$, we have $\sigma = -7$; the equation for B becomes

$$B^3 + 1358525B + 413536578 \quad = 0;$$

that is,

$$(B + 287)(B^2 - 287B + 1440894) = 0,$$

and the roots are

$$-287 \text{ and } \tfrac{1}{2}(287 \pm 497i\sqrt{23}), \text{ or, say } -7.41 \text{ and } \tfrac{7}{2}(41 \pm 71i\sqrt{23}).$$

We have as before, $\sqrt{\delta} = \dfrac{8\sqrt{69}}{9}$, and $\sqrt[3]{\tfrac{1}{2}\rho^2(W + \sqrt{\delta})} = \tfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}} = \theta$; also $\bar{U} = \dfrac{-878}{63}$, whence $\bar{U} + \sqrt{\delta} = \dfrac{2(-439 + 28\sqrt{69})}{63}$. We thus have

$$f' = -\tfrac{9}{10}49 \cdot \frac{2(-439 + 28\sqrt{69})}{63}\tfrac{1}{3}\sqrt[3]{81 + 12\sqrt{69}},$$

$$= -\tfrac{7}{15}(-439 + 28\sqrt{69})\sqrt[3]{81 + 12\sqrt{69}},$$

or, putting for the cube root its value $= \tfrac{1}{2}(3 + \sqrt{69})$, this is

$$f' = -\tfrac{7}{30}(-439 + 28\sqrt{69})(3 + \sqrt{69}), \quad = -\tfrac{287}{2} + \tfrac{497}{6}\sqrt{69}.$$

Similarly, $g' = -\tfrac{287}{2} - \tfrac{497}{6}\sqrt{69}$; and forming the values $f' + g'$, $\omega f' + \omega^2 g'$, $\omega^2 f' + \omega g'$, we have the real root -287 and the imaginary roots $\tfrac{1}{2}(287 \pm 497i\sqrt{23})$, as above.

62. We have the equations

$$B = \bar{L}\theta \quad + \bar{M}\phi,$$

$$A = L\theta \quad + M\phi,$$

$$A^2 - 2LM\theta\phi = \frac{M^2\phi^3}{\theta\phi}\theta + \frac{L^2\theta^3}{\theta\phi}\phi,$$

from which, eliminating θ and ϕ so far as they present themselves linearly on the right-hand side, and in the resulting equation replacing $\theta\phi$ and $LM\theta\phi$ by their values, we have

$$\begin{vmatrix} B, & \bar{L}, & \bar{M} \\ A, & L, & M \\ -\tfrac{5}{3}\rho^2(A^2+\tfrac{2}{3}\rho^2 q_1), & M^2\phi^3, & L^2\theta^3 \end{vmatrix} = 0;$$

that is,

$$B(L^3\theta^3 - M^3\phi^3) = A(L^2\bar{L}\theta^3 - M^2\bar{M}\phi^3) - \tfrac{5}{3}\rho^2(A^2+\tfrac{2}{3}\rho^2 q_1)(L\bar{M} - \bar{L}M).$$

This may be written

$$B\rho^2 t_1\sqrt{\delta} = A\left\{-\tfrac{729}{1000}\frac{49}{\rho^4}[(U-\sqrt{\delta})^2(\bar{U}-\sqrt{\delta})\tfrac{1}{2}\rho^2(W-\sqrt{\delta}) - (U+\sqrt{\delta})^2(\bar{U}+\sqrt{\delta})\tfrac{1}{2}\rho^2(W+\sqrt{\delta})]\right\}$$

$$-\tfrac{5}{3}\rho^2(A^2+\tfrac{2}{3}\rho^2 q_1)\tfrac{81}{100}\frac{49}{\rho^4}[(U-\sqrt{\delta})(\bar{U}+\sqrt{\delta}) - (U+\sqrt{\delta})(\bar{U}-\sqrt{\delta})],$$

where the terms in [ ] contain each of them the factor $\sqrt{\delta}$. Omitting this factor from the equation, and multiplying by ρ^2, we have

$$B\rho^4 t_1 = \tfrac{81}{100}49\{\tfrac{9}{10}A[(U^2+2U\bar{U}+\delta)W + U^2\bar{U} + (2U+\bar{U})\delta] - \tfrac{10}{3}(A^2+\tfrac{2}{3}\rho^2 q_1)(U-\bar{U})\},$$

which I verify at this stage by writing, as before, $\rho=1$. We have $B=-287$, $A=-15$, $t_1=-64$, $q_1=-163$, $W=6$, $U=-\tfrac{34}{9}$, $\bar{U}=\dfrac{-878}{63}$; and, omitting intermediate steps, the equation becomes

$$287\,.\,64 = \frac{81\,.\,49}{100}\left(\tfrac{2496000}{567} - \tfrac{2233600}{567}\right), = \frac{81\,.\,49}{100\,.\,567}262400, = 18368,$$

which is right.

63. We require the values of $(U^2+2U\bar{U}+\delta)W + U^2\bar{U} + (2U+\bar{U})\delta$, and of $U-\bar{U}$: I insert some of the steps of the calculation. We have

$$U^2+2U\bar{U}+\delta = \frac{1}{63^2}\{(133\rho^2-371)(239\rho^2-2233) + 63^2(\rho^4+49) + 3\,.\,49\,.\,122\rho^2\}$$

$$= \frac{1}{63^2}\{35756\rho^4 - 367724\rho^2 + 1022924\}$$

$$= \frac{4}{567}\{1277\rho^4 - 13133\rho^2 + 36533\}.$$

Multiplying by W, $=-\rho^2+7$, we have

$$(U^2+2U\bar{U}+\delta)W = \frac{4}{567}\{-1277\rho^6 + 22072\rho^4 - 128464\rho^2 + 255731\}$$

$$= \frac{4}{5103}\{-11493\rho^6 + 198648\rho^4 - 1156176\rho^2 + 2301579\},$$

$$U^2\bar{U} = \frac{1}{81\,.\,63}(19\rho^2-53)^2(53\rho^2-931) = \frac{1}{5103}\{19133\rho^6 - 442833\rho^4 + 2023911\rho^2 - 2615179\},$$

$$(2U+\overline{U})\,\delta=\frac{1}{63\,.\,27}\,(319\rho^2-1673)\,(27\rho^4+122\rho^2+1323)$$

$$=\frac{1}{1701}\,\{8613\rho^6-6253\rho^4+217931\rho^2-221379\}$$

$$=\frac{1}{5103}\,\{25839\rho^6-\ 18759\rho^4+\ 653793\rho^2-6640137\},$$

whence

$$U^2\overline{U}+(2U+\overline{U})\,\delta=\frac{1}{5103}\,\{44972\rho^6-461592\rho^4+2677704\rho^2-9255316\}$$

$$=\frac{4}{5103}\,\{11243\rho^6-115398\rho^4+\ 669426\rho^2-2313829\}.$$

Hence, adding, we obtain

$$(U^2+2U\overline{U}+\delta)\,W+U^2\overline{U}+(2U+\overline{U})\,\delta$$

$$=\ \frac{4}{5103}\ \{-250\rho^6+83250\rho^4-486750\rho^2-12250\}$$

$$=\frac{-1000}{5103}\,\{\qquad\rho^6-\ \ 333\rho^4+\ \ 1947\rho^2+\qquad 49\}:$$

and we have at once

$$U-\overline{U}=\frac{1}{63}\,(80\rho^2+560)=\frac{80}{63}\,(\rho^2+7).$$

64. We now find

$$B\rho^4t_1=-7A\,(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(3A^2+2\rho^2q_1)\,(\rho^2+7),$$

viz. substituting for t_1, q_1 their values, this is

$$B\rho^4\,(\rho^6+75\rho^4-141\rho^2+1)=-7A\,(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(3A^2+2\rho^2\,(14\rho^4-119\rho^2+1))\,(\rho^2+7),$$

which is the value of B, expressed rationally in terms of ρ, A; it will be observed that B is obtained as a quadric function of A, which is the proper form.

Writing $\rho=-1$, we have $A=-15$, $B=-287$, $t_1=-64$, $q_1=-163$, and the equation is

$$287\,.\,64=105\,.\,1664-56\,.\,349\,.\,8,\ =174720-156352,\ =18368,$$

which is right.

65. Writing for B, A their values $=-\dfrac{56}{\rho}\,\beta-\dfrac{343}{\rho^2}$, and $8\rho\alpha-7\rho^2$, we have

$$\rho^4\left(-\frac{56}{\rho}\,\beta-\frac{343}{\rho^2}\right)t_1=(-56\rho\alpha+49\rho^2)\,(\rho^6-333\rho^4+1947\rho^2+49)$$
$$-56\,(192\rho^2\alpha^2-336\rho^3\alpha+147\rho^4+2\rho^2q_1)\,(\rho^2+7)\,;$$

that is,

$$\begin{aligned}-56\rho^3\beta t_1 = &-56\,.\,192\rho^2(\rho^2+7)\alpha^2\\ &-56\rho\alpha(\rho^6-333\rho^4+1947\rho^2+49)\\ &+56\,.\,336\rho^3\alpha(\rho^2+7)\\ &+49\rho^2(\rho^6-333\rho^4+1947\rho^2+49)\\ &-56\,(147\rho^4+2\rho^2(14\rho^4-119\rho^2-58))(\rho^2+7)\\ &+343\rho^2(\rho^6+75\rho^4-141\rho^2+1),\end{aligned}$$

where the fourth and sixth lines unite into a term divisible by 56, viz. omitting in the first instance a factor 49, the lines are

$$\rho^8-333\rho^6+1947\rho^4+49\rho^2,$$

and

$$7\rho^8+525\rho^6-\ \ 987\rho^4+\ \ 7\rho^2,$$

which together are

$$=8\rho^8+192\rho^6+960\rho^4+56\rho^2,$$

and hence, restoring the factor 49, the lines are

$$=392\,(\rho^8+24\rho^6+120\rho^4+7\rho^2),$$

and the formula now easily becomes

$$\begin{aligned}\rho^2\beta t_1 = &\ 192\rho\,(\rho^2+7)\,\alpha^2\\ &+(\rho^6-669\rho^4-405\rho^2+49)\,\alpha\\ &+\rho\,(21\rho^6-63\rho^4-1593\rho^2-861),\end{aligned}$$

where the last line is

$$=\rho\,(\rho^2+7)\,(21\rho^4-210\rho^2-123).$$

66. Hence, finally, substituting for t_1 its value, we have

$$\beta\rho^2(\rho^6+75\rho^4-141\rho^2+1)=3\rho\,(\rho^2+7)\,(64\alpha^2+7\rho^4-70\rho^2-41)+\alpha\,(\rho^6-669\rho^4-405\rho^2+49),$$

which is the expression for β as a rational function of ρ, α.

Here $\rho=1$, $\alpha=-1$, $\beta=-1$ give $64=-960+1024$, which is right; and again, $\rho=-7$, $\alpha=-1$, $\beta=-1$ give

$$\begin{aligned}-49\,(117649+180075-6909+1)=&-21\,.\,56\,(64+16807-3430-41)\\ &-(117649-1606269-19845+49);\end{aligned}$$

that is,

$$-49\,.\,290816=-1176\,.\,13400+1508416,$$

or,

$$-14249984\ \ =-15758400\quad+1508416,$$

which is right.

The $\alpha\beta$-Differential Equation. Art. No. 67.

67. We have, No. 10,

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{7}\frac{d\alpha}{\alpha^2-1};$$

it should, of course, be possible to verify this equation by means of the $\rho\alpha$-equation and the value just obtained for β. But the expression for $\frac{d\rho}{d\alpha}$ given by the $\rho\alpha$-equation is of so complicated a form that I do not see in what way the verification will come out, and I have not attempted to effect it.

The Coefficients A_1 and A_2. Art. Nos. 68 to 71.

68. These are given by the formulæ No. 47, viz. we have

$$A_1=\frac{1}{\rho}7(\alpha^2-1)\frac{d\rho}{d\alpha}-\tfrac{7}{2}\alpha+\tfrac{1}{2}\beta\rho^2,$$

$$A_2=7(\alpha^2-1)\frac{d\rho}{d\alpha}-\tfrac{19}{6}\alpha\rho+\tfrac{1}{6}\beta\rho^3,$$

where $\frac{d\rho}{d\alpha}$ and β have each of them to be expressed in terms of ρ, α; we have thus A_1 and A_2, each of them expressible rationally in terms of ρ, α; but I have not attempted to effect the substitutions.

69. The five equations of No. 42, merely collecting the terms, are

$$12A_2-6A_1^2-8\alpha A_1+\rho^4-7=0,$$

$$(-6A_1-32\alpha+2\rho^3)A_2-2A_1^3-8A_1+30\rho=0,$$

$$(\rho^2-4)A_2^2+(-4A_1^2-8\alpha A_1+6)A_2-5A_1^2+(2\rho^3+4\rho)A_1-72\alpha\rho=0,$$

$$-2A_1A_2^2+\{(2\rho^2-4)A_1-6\rho\}A_2-4\rho A_1^2-32\rho\alpha A_1+2\rho^3+28\rho=0,$$

$$-3A_2^2+(-4\rho A_1+2\rho^2-8\alpha\rho)A_2+\rho^2A_1^2+10\rho A_1-6\rho^2=0,$$

which would, of course, be all of them satisfied by the values of A_1, A_2 as rational functions of ρ, α, viz. the substitution of these values in any one of the equations would give a function of ρ and α, containing as a factor the expression on the left-hand side of the $\rho\alpha$-equation.

70. Or again, the equations should determine A_1 and A_2 as rational functions of ρ, α, but there is no obvious way of finding such values in a simple form. We, of course, have

$$12A_2=6A_1^2+8\alpha A_1-\rho^4+7,$$

and using this value to eliminate A_2 from the remaining equations, we find the following four equations:

$$A_1^3 . 30 + A_1^2(120\alpha - 6\rho^3) + A_1\{128\alpha^2 - 8\rho^3\alpha - 3\rho^4 + 69\}$$
$$+ \alpha(-16\rho^4 + 112) + \rho^7 - 7\rho^3 - 180\rho = 0,$$

$$A_1^4(36\rho^2 - 432) + A_1^3\alpha(96\rho^2 - 1344)$$
$$+ A_1^2\{\alpha^2(64\rho^2 - 1024) - 12\rho^6 + 48\rho^4 + 84\rho^2 - 624\}$$
$$+ A_1\{\alpha(-16\rho^6 + 160\rho^4 + 112\rho^2 - 544) + 288\rho^3 + 576\rho\}$$
$$+ \{-10368\alpha\rho + \rho^{10} - 4\rho^8 - 14\rho^6 - 16\rho^4 + 49\rho^2 - 308\} = 0,$$

$$A_1^5 . 36 + A_1^4 . 96\alpha + A_1^3\{64\alpha^2 - 12\rho^4 - 72\rho^2 + 208\}$$
$$+ A_1^2\{\alpha(-16\rho^4 - 96\rho^2 + 304) + 504\rho\}$$
$$+ A_1\{\alpha . 2592\rho + \rho^8 - 12\rho^6 + 10\rho^4 + 84\rho^2 - 119\}$$
$$+ 36\rho^5 - 144\rho^3 - 2268\rho = 0,$$

$$A_1^4 . 36 + A_1^3(96\alpha + 96\rho) + A_1^2\{64\alpha^2 + 320\alpha\rho - 12\rho^4 - 96\rho^2 + 84\}$$
$$+ A_1\{256\alpha^2\rho + \alpha(-80\rho^4 + 112) - 16\rho^5 - 368\rho\}$$
$$+ \{\alpha(-32\rho^5 + 224\rho) + \rho^8 + 8\rho^6 - 14\rho^4 + 232\rho^2 + 49\} = 0,$$

and we could from these equations obtain various rational expressions for A_1 and its powers; but these would apparently be of degrees far too high in ρ and α.

71. It is to be remarked that, for $\rho = 1$, $\alpha = -1$, the values of A_1, A_2 are $A_1 = A_2 = 3$, viz. these belong to the solution

$$y = \frac{x(1 + 3x^2 + 3x^4 + x^6)}{1 + 3x^2 + 3x^4 + x^6}, \ = x, \text{ of } \frac{dy}{1 + y^2} = \frac{dx}{1 + x^2};$$

and that for $\rho = -7$, $\alpha = -1$, the values are $A_1 = -21$, $A_2 = 35$, viz. these belong to the solution

$$y = \frac{-7x + 35x^3 - 21x^5 + x^7}{1 - 21x^2 + 35x^4 - 7x^6} \text{ of } \frac{dy}{1 + y^2} = \frac{-7dx}{1 + x^2}.$$

For example, the equation $12A_2 = 6A_1^2 + 8\alpha A_1 - \rho^4 + 7$ becomes, for the first set of values, $36 = 54 - 24 - 1 + 7$, and for the second set of values, $420 = 2646 + 168 - 2401 + 7$, which are each of them right.

New Form of the Seventhic Transformation. Art. Nos. 72 to 83.

72. For the quartic function $1 - 2\alpha x^2 + x^4$, the coefficients a, b, c, d, e are $= 1, 0, -\frac{1}{3}\alpha, 0, 1$, and hence the invariants I, J and the discriminant Δ are

$$I = 1 + \tfrac{1}{3}\alpha^2, \ = \tfrac{1}{3}(\alpha^2 + 3),$$
$$J = -\tfrac{1}{3}\alpha + \tfrac{1}{27}\alpha^3, \ = \tfrac{1}{27}(\alpha^3 - 9\alpha),$$
$$\Delta = I^3 - 27J^2, \ = \tfrac{1}{27}\{(\alpha^2 + 3)^3 - (\alpha^3 - 9\alpha)^2\}, \ = (\alpha^2 - 1)^2, \text{ whence } \sqrt[12]{\Delta} = \sqrt[6]{\alpha^2 - 1}.$$

This being so, then assuming

$$\rho = p\frac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}},$$

the differential equation

$$\frac{dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{\rho\, dx}{\sqrt{1-2\alpha x^2+x^4}}$$

becomes

$$\frac{\sqrt[6]{\beta^2-1}\, dy}{\sqrt{1-2\beta y^2+y^4}} = \frac{p\sqrt[6]{\alpha^2-1}\, dx}{\sqrt{1-2\alpha x^2+x^4}},$$

viz. this is, for the radicals $\sqrt{1-2\alpha x^2+x^4}$ and $\sqrt{1-2\beta y^2+y^4}$, the form considered by Klein in the paper "Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen fünften Grades," *Math. Ann.*, t. XIV. (1879), pp. 111—172. I notice that there is some error as to a factor 7, and that p is equal to the z of p. 148, not, as might appear, equal to $\frac{1}{7}z$.

73. The modular equation presents itself in the form given, *l.c.*, p. 143, viz. this is

$$\mathbf{J} : \mathbf{J}-1 : 1 = (\tau^2+13\tau+49)(\tau^2+5\tau+1)^3 : (\tau^4+14\tau^3+63\tau^2+70\tau-7)^2 : 1728\tau,$$

with the like relation in $\mathbf{J}'$, τ'; and then $\tau\tau'=49$. We have thus $\mathbf{J}$, $\mathbf{J}'$ each given as a function of τ; and thence by elimination of τ, we have the modular equation as a relation between the absolute invariants $\mathbf{J}$, $\mathbf{J}'$. But $\tau = p^2$, and for the form $1-2\alpha x^2+x^4$, as appears above, we have

$$\mathbf{J}-1, = \frac{27J^2}{\Delta};\ = \frac{\frac{1}{27}(\alpha^3-9\alpha)^2}{(\alpha^2-1)^2};$$

hence Klein's equation

$$\mathbf{J}-1 = \frac{(\tau^4+14\tau^3+63\tau^2+70\tau-7)^2}{1728\tau}$$

becomes

$$\frac{\alpha^3-9\alpha}{\alpha^2-1} = \frac{p^8+14p^6+63p^4+70p^2-7}{8p};$$

or, say

$$p^8+14p^6+63p^4+70p^2-8\left(\frac{\alpha^3-9\alpha}{\alpha^2-1}\right)p-7=0,$$

which is the equation, *l.c.*, p. 148 with p for z; viz. this is the $p\alpha$-equation connecting α with the new multiplier p. It will be observed that it is of the degree 8 in p, and the degree 3 in α, viz. it resembles herein the foregoing $\rho\alpha$-equation, but the form is very much more simple, inasmuch as the α enters into a single coefficient only. The equation may also be written

$$(p^4+5p^2+1)^3(p^4+13p^2+49) - 64\frac{(\alpha^2+3)^3}{(\alpha^2-1)^2}p^2 = 0.$$

74. Using for shortness a single letter m to denote the value $i\sqrt{3}$, we have

$$\frac{\alpha^3-9\alpha+3m(\alpha^2-1)}{\alpha^3-9\alpha-3m(\alpha^2-1)} = \frac{p^8+14p^6+63p^4+70p^2+24mp-7}{p^8+14p^6+63p^4+70p^2-24mp-7},$$

that is,

$$\left(\frac{\alpha+m}{\alpha-m}\right)=\frac{(p^2-mp+1)^3(p^2+3mp-7)}{(p^2+mp+1)^3(p^2-3mp-7)},$$

or, say

$$\frac{\alpha+m}{\alpha-m}=\frac{p^2-mp+1}{p^2+mp+1}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

which is another form of the $p\alpha$-equation.

75. We had $\tau=p^2$; and similarly, writing $\tau'=q^2$, then $\tau\tau'=49=p^2q^2$; it must be assumed that $pq=-7$; β is then the same function of q which α is of p, viz. we have

$$\frac{\beta+m}{\beta-m}=\frac{q^2-mq+1}{q^2+mq+1}\sqrt[3]{\frac{q^2+3mq-7}{q^2-3mq-7}}.$$

These equations in α and β contain the same cubic radical, viz. we have

$$q^2+3mq-7,\ =\frac{49}{p^2}-\frac{21m}{p}-7,\ =-\frac{7}{p^2}(p^2+3mp-7),$$

and similarly

$$q^2-3mq-7\qquad\qquad =-\frac{7}{p^2}(p^2-3mp-7).$$

Moreover

$$q^2-\ mq+1,\ =\frac{49}{p^2}+\frac{7m}{p}+1,\ =\frac{1}{p^2}(p^2+7mp+49),$$

and similarly

$$q^2+\ mq+1\qquad\qquad =\frac{1}{p^2}(p^2-7mp+49),$$

and we thus obtain

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\sqrt[3]{\frac{p^2+3mp-7}{p^2-3mp-7}},$$

whence, eliminating the cubic radical,

$$\frac{\beta+m}{\beta-m}=\frac{p^2+7mp+49}{p^2-7mp+49}\,\frac{p^2+mp+1}{p^2-mp+1}\,\frac{\alpha+m}{\alpha-m},$$

viz. this gives β as a rational function of α, p. We in fact have

$$\beta=\frac{\alpha(p^4+29p^2+49)-24p(p^2+7)}{8\alpha p(p^2+7)+(p^4+29p^2+49)}.$$

76. The differential relation

$$\frac{d\beta}{\beta^2-1}=\frac{\rho^2}{7}\,\frac{d\alpha}{\alpha^2-1},$$

on substituting therein for ρ its value, becomes

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{p^2}{7}\,\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}.$$

But, from the expression for $\dfrac{\alpha+m}{\alpha-m}$, we obtain

$$d\alpha\left(\frac{1}{\alpha+m}-\frac{1}{\alpha-m}\right)=dp\left\{\left(\frac{2p-m}{p^2-mp+1}-\frac{2p+m}{p^2+mp+1}\right)+\tfrac{1}{3}\left(\frac{2p+3m}{p^2+3mp-7}-\frac{2p-3m}{p^2-3mp-7}\right)\right\},$$

or, omitting from each side a factor $-2m$,

$$\frac{d\alpha}{\alpha^2+3}=dp\left(\frac{-p^2+1}{p^4+5p^2+1}+\frac{p^2+7}{p^4+13p^2+49}\right)=\frac{56dp}{(p^4+5p^2+1)(p^4+13p^2+49)}.$$

But we have, No. 73,

$$\frac{\alpha^2+3}{(\alpha^2-1)^{\frac{2}{3}}}=\frac{(p^4+5p^2+1)(p^4+13p^2+49)^{\frac{1}{3}}}{4p^{\frac{2}{3}}},$$

and thence

$$\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}=\frac{14dp}{p^{\frac{2}{3}}(p^4+13p^2+49)^{\frac{2}{3}}},$$

and similarly

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{14dq}{q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}}.$$

The equation $q=-\dfrac{7}{p}$ gives

$$dq=\frac{7dp}{p^2},\quad q^{\frac{2}{3}}(q^4+13q^2+49)^{\frac{2}{3}}=49p^{-\frac{10}{3}}(p^4+13p^2+49)^{\frac{2}{3}},$$

and we thence have

$$\frac{d\beta}{(\beta^2-1)^{\frac{2}{3}}}=\frac{2p^{\frac{4}{3}}\,dp}{(p^4+13p^2+49)^{\frac{2}{3}}},\ =\frac{p^2}{7}\,\frac{d\alpha}{(\alpha^2-1)^{\frac{2}{3}}},$$

the required relation.

77. From the value of ρ, we have

$$\frac{d\rho}{\rho}=\frac{dp}{p}+\frac{\frac{1}{3}\alpha d\alpha}{\alpha^2-1}-\frac{\frac{1}{3}\beta d\beta}{\beta^2-1},$$

which, substituting for $d\beta$ its value, becomes

$$=\frac{dp}{p}+\frac{\frac{1}{3}d\alpha}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\,\frac{p^2}{7}\right\},$$

or, say

$$\frac{1}{\rho}\,\frac{d\rho}{d\alpha}=\frac{1}{p}\,\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{(\alpha^2-1)^{\frac{2}{3}}}\left\{\frac{\alpha}{(\alpha^2-1)^{\frac{1}{3}}}-\frac{\beta}{(\beta^2-1)^{\frac{1}{3}}}\,\frac{p^2}{7}\right\},$$

which, however, is more conveniently written

$$\frac{1}{\rho}\,\frac{d\rho}{d\alpha}=\frac{1}{p}\,\frac{dp}{d\alpha}+\frac{\frac{1}{3}}{\alpha^2-1}(\alpha-\beta\rho^2);$$

and then, substituting in the formulæ for A_1, A_2, we find

$$A_1 = 7\,(\alpha^2-1)\,\frac{1}{p}\,\frac{dp}{d\alpha} - \tfrac{7}{6}\alpha + \tfrac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}A_2 = 7\,(\alpha^2-1)\,\frac{1}{p}\,\frac{dp}{d\alpha} - \tfrac{5}{6}\alpha - \tfrac{1}{6}\beta\rho^2,$$

expressions which give, as they should do, $A_2 - \rho A_1 = \tfrac{1}{3}(\alpha\rho - \beta\rho^3)$. In these last formulæ, ρ is to be regarded as standing for its value, $= p\,\dfrac{\sqrt[6]{\alpha^2-1}}{\sqrt[6]{\beta^2-1}}$.

78. To further reduce these values, consider the expression of β given No. 75. If for a moment we represent this by

$$\beta = \frac{F\alpha - 3G}{G\alpha + F}, \text{ where } F = p^4 + 29p^2 + 49,\; G = 8p\,(p^2+7),$$

then we have

$$\beta^2 - 1 = \frac{(F^2-G^2)\,\alpha^2 - 8FG\alpha + 9G^2 - F^2}{(G\alpha+F)^2},$$

or, multiplying the numerator and denominator each by $G\alpha + F$, so as to make the denominator a perfect cube, the numerator becomes

$$G\,(F^2-G^2)\,(\alpha^3-9\alpha) + F\,(F^2-9G^2)\,(\alpha^2-1);$$

and putting for the factor G of the first term its value $= 8p\,(p^2+7)$, we thus obtain

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(F^2-G^2)\,(p^2+7)\,8p\left(\dfrac{\alpha^3-9\alpha}{\alpha^2-1}\right) + F\,(F^2-9G^2)}{(G\alpha+F)^3},$$

viz. in virtue of the $p\alpha$-equation, this is

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(F^2-G^2)\,(p^2+7)\,(p^8+14p^6+63p^4+70p^2-7) + F\,(F^2-9G^2)}{(G\alpha+F)^3}.$$

This numerator is $=(p^4+5p^2+1)^3\,p^6$; in fact, we have

$$\begin{aligned}(F^2-G^2)\,(p^2+7) &= p^{10} + p^8 + p^6 + 7p^4 + 343p^2 + 16807,\\ F^2 - 9G^2 &= p^8 - 518p^6 - 7125p^4 - 25382p^2 + 2401,\end{aligned}$$

and thence forming the two terms of the numerator and adding them together—for shortness I write down only the coefficients—we have

	1	15	78	154	567	22113	257390	1082802	1174089	− 117649
				1	− 489	− 22098	− 257389	− 1082802	− 1174089	117649
=	1	15	78	155	78	15	1	0	0	0,

viz. these are the coefficients of $(p^4+5p^2+1)^3\,p^6$. Hence

$$\frac{\beta^2-1}{\alpha^2-1} = \frac{(p^4+5p^2+1)^3\,p^6}{(G\alpha+F)^3};$$

or, extracting the cube root, and for G, F substituting their values,

$$\frac{\sqrt[3]{\beta^2-1}}{\sqrt[3]{\alpha^2-1}}=\frac{(p^4+5p^2+1)\,p^2}{8p\,(p^2+7)\,\alpha+p^4+29p^2+49},$$

and thence also

$$\rho^2=\frac{8p\,(p^2+7)\,\alpha+p^4+29p^2+49}{p^4+5p^2+1},$$

viz. we have thus ρ^2 expressed as a rational function of p, α.

79. It will presently appear that ρ is, in fact, expressible as a rational function of p, α: but I am unable to obtain this expression in a simple form. Admitting that ρ is thus expressible, a direct process for obtaining the expression is as follows. Writing

$$\xi=\frac{8p\,(p^2+7)\,\alpha+p^4+29p^2+49}{p^4+5p^2+1}\,(=\rho^2),$$

and by means hereof introducing ξ in place of α into the equation

$$p^8+14p^6+63p^4+70p^2-8p\,\frac{\alpha^3-9\alpha}{\alpha^2-1}-7=0,$$

we have for ξ a cubic equation,

$$a\xi^3+b\xi^2+c\xi+d=0,$$

where the coefficients a, b, c, d are given rational functions of p. This equation may be written

$$a\xi\,(\xi+\vartheta)^2+b'\xi^2+c'\xi+d=0,$$

where $b'=b-2a\vartheta$, $c'=c-a\vartheta^2$; and the last three terms will be a square if only $c'^2-4b'd=0$; that is, if

$$(a\vartheta^2-c)^2+4d\,(2a\vartheta-b)=0,$$

a biquadratic equation in ϑ which (ρ being expressible as above) must have one of its roots = a rational function of p. Calling this ϑ, we then have

$$a\xi\,(\xi+\vartheta)^2+\frac{1}{b'}\,(b'\xi+\tfrac{1}{2}c')^2=0,\ \text{or say}\ a\rho^2\,(\xi+\vartheta)^2+\frac{1}{b'}\,(b'\xi+\tfrac{1}{2}c')^2=0,$$

hence

$$\rho=\sqrt{\frac{-1}{ab'}}\cdot\frac{b'\xi+\tfrac{1}{2}c'}{\xi+\vartheta},$$

where ξ denotes a linear function of α as above; the quadric radical will have a rational value, and the form of the equation thus is

$$\rho=\frac{A\alpha+B}{C\alpha+D},$$

where A, B, C, D are rational and integral functions of p. But I am not able to carry out the process.

80. As shown, No. 78, we have

$$\rho^2 = \frac{8p\,(p^2+7)\,\alpha + p^4 + 29p^2 + 49}{p^4 + 5p^2 + 1}.$$

Multiplying by the value of β, *ante* No. 75, we find

$$\beta\rho^2 = \frac{(p^4 + 29p^2 + 49)\,\alpha - 24p\,(p^2+7)}{p^4 + 5p^2 + 1};$$

and we can hence find A_1 and A_2 by the formulæ

$$A_1 = 7\,(\alpha^2 - 1)\,\frac{1}{p}\,\frac{dp}{d\alpha} - \tfrac{7}{6}\alpha + \tfrac{1}{6}\beta\rho^2,$$

$$\frac{1}{\rho}\,A_2 = 7\,(\alpha^2 - 1)\,\frac{1}{p}\,\frac{dp}{d\alpha} - \tfrac{5}{6}\alpha - \tfrac{1}{6}\beta\rho^2,$$

or, for the second of these we may write

$$\frac{1}{\rho}\,A_2 = A_1 + \tfrac{1}{3}(\alpha - \beta\rho^2).$$

But in a different point of view, regarding only ρ^2, but not ρ, as a given function of p, α, we must to these equations join the equation $12A_2 = 6A_1{}^2 + 8\alpha A_1 - \rho^4 + 7$, *ante* No. 69: and we have thus equations for the determination of A_1, A_2, and ρ.

81. We have

$$\begin{aligned} A_1 = {} & \frac{(p^4 + 5p^2 + 1)(p^4 + 13p^2 + 49)}{8p}\,\frac{\alpha^2 - 1}{\alpha^2 + 3} \\ & - \tfrac{7}{6}\alpha + \frac{\alpha\,(p^4 + 29p^2 + 49) - 24p\,(p^2+7)}{6\,(p^4 + 5p^2 + 1)}, \end{aligned}$$

where the second line is

$$= \frac{\alpha\,(-p^4 - p^2 + 7) - 4p\,(p^2+7)}{p^4 + 5p^2 + 1}.$$

Uniting the two terms, we have a denominator $8p\,(p^4 + 5p^2 + 1)$, and in the numerator a term $8p\alpha^3$ which may be got rid of by means of the $p\alpha$-equation; the numerator thus becomes

$$\begin{aligned} & = 96p\,(-p^4 - p^2 + 7) - 128p^2\,(p^2+7)\,\alpha \\ & \quad + (\alpha^2 - 1)\,\{(-p^4 - p^2 + 7)(p^8 + 14p^6 + 63p^4 + 70p^2 - 7)\} \\ & \quad + (p^4 + 5p^2 + 1)^2\,(p^4 + 13p^2 + 49) - 32p^2\,(p^2+7), \end{aligned}$$

where the whole divides by $8p$; and we finally obtain

$$A_1 = \frac{12\,(-p^4 - p^2 + 7) - 16p\,(p^2+7)\,\alpha + (\alpha^2 - 1)\,p\,(p^8 + 17p^6 + 102p^4 + 225p^2 + 97)}{(\alpha^2 + 3)\,(p^4 + 5p^2 + 1)}.$$

Proceeding to calculate the value of $A_1 + \frac{1}{3}(\alpha - \beta\rho^2)$, we then have

$$\tfrac{1}{3}(\alpha - \beta\rho^2) = \frac{-8\,(p^2+2)\,\alpha + 8p\,(p^2+7)}{p^4 + 5p^2 + 1}.$$

Multiplying the numerator and denominator by α^2+3, we have in the numerator a term in $8\alpha^3$, which may be got rid of by means of the $p\alpha$-equation; the numerator for $\frac{1}{\rho}A_2$ thus becomes

$$12(-p^4-9p^2-9)+16p(p^2+7)\alpha$$
$$+(\alpha^2-1)p\left[\{p^8+17p^6+102p^4+225p^2+97\}-\frac{p^2+2}{p^2}(p^8+14p^6+63p^4+70p^2-7)+8(p^2+7)\right],$$

and we finally obtain

$$\frac{1}{\rho}A_2=\frac{12(-p^4-9p^2-9)+16p(p^2+7)\alpha+(\alpha^2-1)p^{-1}(p^8+11p^6+37p^4+20p^2+2)}{(\alpha^2+3)(p^4+5p^2+1)}.$$

82. The expressions obtained above for ρ^2, A_1, A_2 are of the form

$$\rho^2=\frac{M+N\alpha}{S},\quad A_1=\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)},\quad \frac{1}{\rho}A_2=\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)},$$

where

$$M=p^4+29p^2+49;\qquad N=8p(p^2+7);\qquad S=p^4+5p^2+1,$$
$$P_1=12(-p^4-p^2+7)-p(p^8+17p^6+102p^4+225p^2+97),\quad Q_1=-16p(p^2+7),$$
$$R_1=p(p^8+17p^6+102p^4+225p^2+97);$$
$$P_2=12(-p^4-9p^2-9)-p^{-1}(p^8+11p^6+37p^4+20p^2+2),\quad Q_2=16p(p^2+7),$$
$$R_2=p^{-1}(p^8+11p^6+37p^4+20p^2+2);$$

substituting these values in the foregoing equation

$$12A_2=6A_1^2+8\alpha A_1-\rho^4+7,$$

we obtain

$$12\rho\left\{\frac{P_2+Q_2\alpha+R_2\alpha^2}{S(\alpha^2+3)}\right\}=\left\{\frac{6(P_1+Q_1\alpha+R_1\alpha^2)^2}{S^2(\alpha^2+3)^2}+8\alpha\frac{P_1+Q_1\alpha+R_1\alpha^2}{S(\alpha^2+3)}-\frac{(M+N\alpha)^2}{S^2}+7\right\};$$

that is,

$$\rho=\frac{1}{12(P_2+Q_2\alpha+R_2\alpha^2)^2(3+\alpha^2)S}\{6(P_1+Q_1\alpha+R_1\alpha^2)^2+8\alpha S(3+\alpha^2)(P_1+Q_1\alpha+R_1\alpha^2)$$
$$-(M+N\alpha)^2(3+\alpha^2)^2+7S^2(3+\alpha^2)^2\},$$

which, by means of the $p\alpha$-equation

$$p^8+14p^6+63p^4+70p^2-\left(\frac{\alpha^3-9\alpha}{\alpha^2-1}\right)8p-7=0,$$

should be reducible to the form

$$\rho=A\alpha^2+B\alpha+C,\ \text{or}\ \rho=\frac{A\alpha+B}{C\alpha+D};$$

but I have not been able to obtain, in either of these forms, a simple expression of ρ as a function of p, α. Supposing it obtained, the $\rho\alpha$-equation, *ante* No. 51, would of course be thereby transformable into the foregoing $p\alpha$-equation. And considering p as an auxiliary parameter thus introduced into the formulæ in place of ρ, then β and the coefficients A_1, A_2 are, by what precedes, expressed in terms of p, α, that is, in effect in terms of ρ, α; and we thus have the formulæ of transformation for the $\rho\alpha\beta$-form.

83. There exists a remarkably simple particular case. Write for convenience $\theta = \sqrt{7}$, the $p\alpha$-equation is satisfied by the values $p = -\theta$, $\alpha = -\frac{3}{8}\theta$. In fact, these values give $8p\alpha = 3\theta^2$, $= 21$, $\frac{\alpha^2 - 9}{\alpha^2 - 1} = (\frac{63}{64} - 9) \div (\frac{63}{64} - 1)$, $= 513$; the term in α is thus $21 . 513$, $= 10773$; but, assuming $p^2 = 7$, we have

$$p^8 + 14p^6 + 63p^4 + 70p^2 - 7 = 2401 + 4802 + 3087 + 490 - 7, \ = 10773,$$

and the equation is thus satisfied. And these values, $p = -\theta$, $\alpha = -\frac{3}{8}\theta$, give $\rho^2 = 7$, $\beta = \frac{3}{8}\theta$, $A_1 = 2\theta$, $A_2 = \rho\theta$; the equation $12A_2 = 6A_1{}^2 + 8\alpha A_1 - \rho^4 + 7$ thus becomes $12\rho\theta = 168 - 42 - 49 + 7$, $= 84$; that is, $\rho\theta = 7$, $= \theta^2$, or $\rho = \theta\,(= -p)$. We have $\alpha^2 - 1$, $= \beta^2 - 1$, $= -\frac{1}{64}$; but from the equation $\rho = p\frac{\sqrt[6]{\alpha^2 - 1}}{\sqrt[6]{\beta^2 - 1}}$, it appears that the sixth roots must be equal with opposite signs, say $\sqrt[6]{\alpha^2 - 1} = \frac{i}{2}$, $\sqrt[6]{\beta^2 - 1} = \frac{-i}{2}$. Retaining θ to stand for its value $= \sqrt{7}$, the differential equation is

$$\frac{dy}{\sqrt{1 - \frac{3}{4}\theta y^2 + y^4}} = \frac{\theta dx}{\sqrt{1 + \frac{3}{4}\theta x^2 + x^4}},$$

and it is satisfied by

$$y = \frac{x\,(\theta + 7x^2 + 2\theta x^4 + x^6)}{1 + 2\theta x^2 + 7x^4 + \theta x^6}.$$

It may be remarked that the quartic functions of y and x resolved into their linear factors are

$$\left\{y + \frac{3i + \theta}{2\sqrt{2}\,(1 + i)}\right\} \left\{y + \frac{3i - \theta}{2\sqrt{2}\,(1 + i)}\right\} \left\{y + \frac{-3i + \theta}{2\sqrt{2}\,(1 - i)}\right\} \left\{y + \frac{-3i - \theta}{2\sqrt{2}\,(1 - i)}\right\},$$

and

$$\left\{x + \frac{3 - i\theta}{2\sqrt{2}\,(1 + i)}\right\} \left\{x + \frac{3 + i\theta}{2\sqrt{2}\,(1 + i)}\right\} \left\{x + \frac{3 - i\theta}{2\sqrt{2}\,(1 - i)}\right\} \left\{x + \frac{3 + i\theta}{2\sqrt{2}\,(1 - i)}\right\}$$

and that for the first of the y-factors, substituting for y its value, we have

$$x^7 + 2\theta x^5 + 7x^3 + \theta x + \frac{3i + \theta}{2\sqrt{2}\,(1 + i)}(\theta x^6 + 7x^4 + 2\theta x^2 + 1)$$
$$= \left\{x + \frac{3 - i\theta}{2\sqrt{2}\,(1 + i)}\right\} \left\{x^3 + \frac{1 + i\theta}{\sqrt{2}\,(1 + i)}x^2 + \tfrac{1}{2}(i + \theta)\,x + \frac{1 + i}{\sqrt{2}}\right\}^2,$$

with like expressions for the other y-factors respectively.

Brioschi's Transformation Theory. Art. No. 84.

84. M. Brioschi has kindly referred me to two papers by him, "Sur une Formule de Transformation des Fonctions Elliptiques," *Comptes Rendus*, t. LXXIX. (1874), pp. 1065—1069, and *ibid.* t. LXXX. (1875), pp. 261—264. They relate to the form

$$\frac{dx}{\sqrt{4x^3 - g_2 x - g_3}} = \frac{dy}{\sqrt{4y^3 - G_2 y - G_3}},$$

with a formula of transformation

$$y = \frac{U}{T^2}, \quad T = x^\nu + a_1 x^{\nu-1} + a_2 x^{\nu-2} + \dots + a_\nu, \text{ where } \nu = \tfrac{1}{2}(n-1),$$

$$U = x^n + \alpha_1 x^{n-2} + \alpha_2 x^{n-3} + \dots + \alpha_\nu.$$

The general theory for any value of n is developed to a considerable extent, and it would without doubt give very interesting results for the case $n = 7$; but the formulæ are only completely worked out for the preceding two cases $n = 3$ and $n = 5$. For these cases the formulæ are as follows:

Cubic transformation: $n = 3$,

$$y = \frac{x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3}{(x + a_1)^2}.$$

Corresponding to the modular equation, we have

$$a_1^4 - \tfrac{1}{2} g_2 a_1^2 + g_3 a_1 - \tfrac{1}{48} g_2^2 = 0,$$

and then

$$G_2 - 9g_2 = 6(20a_1^2 - 3g_2), \quad G_3 + 27g_3 = -14(20a_1^2 - 3g_2)\, a_1,$$

whence also

$$a_1 = -\tfrac{3}{7} \frac{G_3 + 27g_3}{G_2 - 9g_2};$$

and by the general theory α_1, α_2, α_3 are given rationally in terms of a_1, g_2, g_3.

Quintic transformation: $n = 5$,

$$y = \frac{x^5 + \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5}{(x^2 + a_1 x + a_2)^2}.$$

We have

$$a_1 X - 2Y = 0, \quad (12a_1^2 + g_2) X - 30 a_1 Y = 0,$$

where

$$X = a_1^3 - 6a_1^2 a_2 + \tfrac{1}{2} g_2 a_1 - g_3,$$

$$Y = 5a_2^2 - a_1^2 a_2 + \tfrac{1}{2} g_2 a_2 - g_3 a_1 + \tfrac{1}{16} g_2^2.$$

The first of these gives

$$a_2 = \frac{1}{6a_1}(a_1^3 + \tfrac{1}{2} g_2 a_1 - g_3);$$

then eliminating a_2, we have, corresponding to the modular equation,

$$a_1^6 - 5g_2 a_1^4 + 40 g_3 a_1^3 - 5g_2^2 a_1^2 + 8 g_2 g_3 a_1 - 5 g_3^2 = 0.$$

We then have

$$G_2 - 25g_2 = \frac{8}{a_1}(10a_1^3 - 8g_2 a_1 + 5g_3), \quad G_3 + 125 g_3 = -14(10a_1^3 - 8g_2 a_1 + 5g_3);$$

whence also

$$a_1 = -\tfrac{4}{7} \frac{G_3 + 125g_3}{G_2 - 25g_2};$$

and by the general theory α_1, α_2, α_3, α_4, α_5 are given rationally in terms of a_1, g_2, g_3.

These results are contained in the former of the papers above referred to; the latter contains some properties of these modular equations.