868.

ON THE INTERSECTION OF CURVES.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 85—90.]

It is only recently that I have studied Bacharach's paper "Ueber den Cayley'schen Schnittpunktsatz," *Math. Ann.*, t. XXVI. (1885), pp. 275—299: his theorem in regard to the case where the δ points lie on a curve of the order $\gamma-3$ is a very interesting and valuable one, but I consider it rather as an addition than as a correction to my original theorem; and I cannot by any means agree that the method by counting of constants is to be rejected as not trustworthy; on the contrary, it seems to me to be the proper foundation of the whole theory; it must of course be employed with due consideration of special cases. I reproduce the theorem in what appears to me the complete form.

Writing with Bacharach

$$r \geqq m,\ \ r \geqq n,\ \ r \lesseqgtr m+n-3,\ \ \gamma = m+n-r,\ \ \delta = \tfrac{1}{2}(\gamma-1)(\gamma-2),$$

and assuming $n \geqq m$, (these equations and inequalities are to be attended to throughout the present paper), I consider two curves of the orders m, n respectively meeting in δ points B, and in $(mn-\delta)$ points A; and I state the theorem as follows:

1°. The $mn-\delta$ points A are in general such that a curve of the order r, which passes through $mn-\delta-1$ of these points, does not of necessity pass through the remaining point; and in this case the general curve of the order r, which passes through the $mn-\delta$ points A, has for its form of equation

$$0 = L_{r-m}P_m + M_{r-n}Q_n,$$

where $P_m=0$, $Q_n=0$ are the equations of the given curves and L_{r-m}, M_{r-n} denote functions of the orders $r-m$, $r-n$ respectively; and it thus appears that the curve of the order r through the $mn-\delta$ points A passes also through the δ points B.

2°. If however the δ points B are on a curve of the order $\gamma - 3$, then the $mn - \delta$ points A are a system such that every curve of the order r passing through $mn - \delta - 1$ of these points passes through the remaining point; and in this case the general curve of the order r, which passes through the $mn - \delta$ points A, has for its form of equation

$$0 = \Omega_r + L_{r-m} P_m + M_{r-n} Q_n,$$

$\Omega_r = 0$ is a particular curve through the $mn - \delta$ points A, which does not go through any of the points B; and consequently the curve of the order r does not pass through any of the points B.

For the proof of the theorem I premise as follows:

A curve of the order r depends upon $\frac{1}{2} r(r+3)$ constants, or to use a convenient expression, its Postulandum is $= \frac{1}{2} r(r+3)$: if the curve has to pass through a given point, this imposes a single relation upon the constants, or say the Postulation is $= 1$; similarly, if the curve has to pass through k given points, this imposes k relations, or say the Postulation is $= k$. The points may be however a special system, for instance, they may be such that every curve of the order r which passes through $k - 1$ of the points, will pass through the remaining point; the Postulation is in this case $= k - 1$; and so in other cases. Assuming the Postulation of the k points to be $= k$, then the Postulandum of a curve of the order r through the k points is $= \frac{1}{2} r(r+3) - k$. I stop to remark that the Postulation has reference to the particular curve or other entity in question; thus in the case of a curve passing through k points, the Postulation for a curve of a certain order may be $= k$, and for a curve of a different order it may be less than k.

Considering now, as above, two given curves of the orders m and n intersecting in the $mn - \delta$ points A and the δ points B, then assuming that the $mn - \delta$ points are not a special system, viz. that their Postulation in regard to a curve of the order r is $= mn - \delta$, the Postulandum of a curve of the order r through the $mn - \delta$ points is

$$= \tfrac{1}{2} r(r+3) - mn + \delta,$$

which is

$$= \tfrac{1}{2}(r - m + 1)(r - m + 2) + \tfrac{1}{2}(r - n + 1)(r - n + 2) - 1,$$

viz. this is identically true when for δ we write its value

$$= \tfrac{1}{2}(m + n - r - 1)(m + n - r - 2).$$

But we have through the $mn - \delta$ points A, a curve

$$L_{r-m} P_m + M_{r-n} Q_n = 0$$

with the proper Postulandum: viz. L_{r-m} contains $\frac{1}{2}(r - m + 1)(r - m + 2)$ constants, M_{r-n} contains $\frac{1}{2}(r - n + 1)(r - n + 2)$ constants, and there is a diminution -1 for the constant which divides out; hence this is the general equation of the curve of the order r through the $mn - \delta$ points A; and the curve passes through the remaining δ points B.

In the case where the δ points B are on a curve of the order $\gamma - 3$, (observe that this is a single condition imposed on the δ, $= \frac{1}{2}(\gamma - 1)(\gamma - 2)$ points, for a curve of the order $(\gamma - 3)$ can be drawn through $\frac{1}{2}\gamma(\gamma - 3)$ points), it is to be shown that the Postulation of the $mn - \delta$ points A is $= mn - \delta - 1$; for, this being so, the Postulandum of the curve of the order r through the $(mn - \delta)$ points A will be

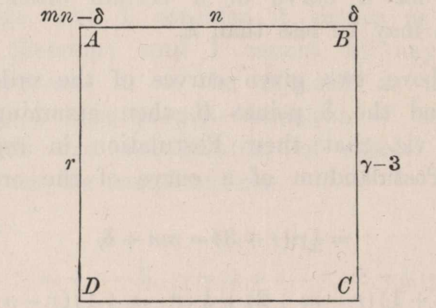
$$= \tfrac{1}{2}r(r+3) - mn + \delta + 1,$$

and the equation of the curve will no longer be of the foregoing form, but it will be of the form

$$\Omega_r + L_{r-m}P_m + M_{r-n}Q_n = 0,$$

$\Omega_r = 0$ being a particular curve through the $mn - \delta$ points A, which does not pass through any of the points B. The proof depends on the theory of Residuation: which for the present purpose may be presented under the following form.

Let A, B, ... denote systems of points upon a given Basis-curve, for instance the foregoing curve $P_m = 0$, of the order m. And let $A = ci$ denote that the points A are the complete intersection of the basis-curve by some other curve; (this implies that the number of the points is $= km$, a multiple of m, and the intersecting curve is then of course a curve of the order k). It is clear that, if $A = ci$, and $B = ci$, then also $A + B = ci$. But conversely we have the theorem that, if $A + B = ci$ and $A = ci$, then also $B = ci$. And we at once deduce the further theorem: if $A + B = ci$, $B + C = ci$, $C + D = ci$, then also $A + D = ci$. For the first and third relations give $A + B + C + D = ci$, and the second relation then gives $A + D = ci$.



Starting now (see the diagram) with the δ points B on a curve of the order $\gamma - 3$, suppose that we have through these points the basis-curve $P_m = 0$ of the order m, and another given curve $Q_n = 0$, of the order n; and let these besides meet in the $mn - \delta$ points A. Let the curve of the order $\gamma - 3$ besides meet the basis-curve in the $m(\gamma - 3) - \delta$ points C; and through these let there be drawn a curve of the order $m - 3$, which besides meets the basis-curve in the $m(r - n) + \delta$ points D. We have here $A + B = ci$, $B + C = ci$, $C + D = ci$; consequently $A + D = ci$, that is, the $mn - \delta$ points A and the $m(r - n) + \delta$ points D lie on a curve of the order r. The curve of the order $m - 3$ passes through the $m(\gamma - 3) - \delta$ points C; its Postulandum is thus

$$= \tfrac{1}{2}m(m-3) - m(\gamma - 3) + \delta,$$

which is

$$= \tfrac{1}{2}(r-n+1)(r-n+2).$$

In fact, substituting for γ, δ their values $= m+n-r$, and $\frac{1}{2}(m+n-r-1)(m+n-r-2)$ respectively, this equation is satisfied identically. The system of the $m(r-n)+\delta$ points D thus depends upon $\frac{1}{2}(r-n+1)(r-n+2)$ constants, or say the Postulandum of the points D is $= \frac{1}{2}(r-n+1)(r-n+2)$. It follows that the curve of the order r through the $(mn-\delta)$ points A and the $m(r-n)+\delta$ points D *cannot* have an equation of the form

$$L_{r-m}P_m + M_{r-n}Q_n = 0;$$

for the intersections of this curve with the basis-curve $P_m = 0$ are given by the equation $M_{r-n}Q_n = 0$, which contains only the

$$\tfrac{1}{2}(r-n+1)(r-n+2)-1$$

constants of M_{r-n} (one constant of course divides out, giving the diminution -1). The equation must have the more general form

$$\Omega_r + L_{r-m}P_m + M_{r-n}Q_n = 0;$$

and it thus appears that the Postulation of the $mn-\delta$ points A, instead of being $= mn-\delta$, must be $= mn-\delta-1$. This completes the proof.

I notice that, combining the last-mentioned identity

$$\tfrac{1}{2}m(m-3) - m(\gamma-3) + \delta = \tfrac{1}{2}(r-n+1)(r-n+2)$$

with the like identity

$$\tfrac{1}{2}n(n-3) - n(\gamma-3) + \delta = \tfrac{1}{2}(r-m+1)(r-m+2),$$

we obtain

$$\tfrac{1}{2}m(m-3) + \tfrac{1}{2}n(n-3) - (m+n)(\gamma-3) + 2\delta$$
$$= \tfrac{1}{2}(r-n+1)(r-n+2) + \tfrac{1}{2}(r-m+1)(r-m+2),$$

and consequently, referring to a former result, the left-hand side should be

$$= \tfrac{1}{2}r(r+3) - mn + \delta + 1;$$

substituting for γ, δ their values, this is at once verified.

As appears by what precedes, Bacharach's special case is that in which the δ, $= \frac{1}{2}(\gamma-1)(\gamma-2)$ points B satisfy the single condition of lying on a curve of the order $\gamma-3$. We may have between the points B more than a single relation; in particular, the points B may be such as to include among themselves the complete intersection of two curves of the orders a, b respectively ($ab \gtreqless \delta$): this will be the case, if the given curves are of the form

$$P_m = \lambda_a S_{m-a} - \mu_b R_{m-b},$$
$$Q_n = \lambda_a V_{n-a} - \mu_b U_{n-b},$$

it being understood, here and in what follows, that the values of a, b are such that the suffixes are none of them negative.

The two curves here intersect in the ab points ($\lambda_a = 0$, $\mu_b = 0$), and in the $mn - ab$ points

$$\left\| \begin{matrix} \lambda_a, & R_{m-b}, & U_{n-b} \\ \mu_b, & S_{m-a}, & V_{n-a} \end{matrix} \right\| = 0,$$

say the $(mn - \delta)$ points A are $mn - ab - \Theta$ of the last-mentioned points, and the δ points B are the remaining Θ points together with the ab points. Here the general form of the curve of the order r passing through the $mn - ab$ points, and therefore through the $mn - \delta$ points A, is

$$\begin{vmatrix} L_{r-m-n+a+b}, & M_{r-n}, & N_{r-m} \\ \lambda_a & , R_{m-b}, & U_{n-b} \\ \mu_b & , S_{m-a}, & V_{n-a} \end{vmatrix} = 0,$$

where $L_{r-m-n+a+b}$, M_{r-n}, N_{r-m} are arbitrary functions of the orders indicated by the respective suffixes. The theory in regard to the number of constants is of course altogether different from that which belongs to the case of the general functions P_m, Q_n; and it is probable that much interesting theory would present itself in the consideration of particular cases.

Cambridge, 22 *March* 1887.