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NOTE ON KIEPERT'S *L*-EQUATIONS, IN THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

[From the *Mathematische Annalen*, t. XXX. (1887), pp. 75—77.]

It appears, by comparison with Klein's paper "Ueber die Transformation u. s. w.," *Math. Annalen*, t. XIV. (1878), see p. 144, that Kiepert's L made use of in the Memoir "Ueber Theilung und Transformation der elliptischen Functionen," *Math. Annalen*, t. XXVI. (1886), pp. 369—454, is, in fact, the square of the multiplier, "für das durch $\sqrt[12]{\Delta}$ normirte Integral," viz. considering the general quartic function $(a, \ldots)(x, 1)^4 = (a, b, c, d, e)(x, 1)^4$, and the transformed function $(a_1, \ldots)(y, 1)^4$, then we have

$$\frac{L^2 \sqrt[12]{\Delta}\,dx}{\sqrt{(a, \ldots)(x, 1)^4}} = \frac{\sqrt[12]{\Delta_1}\,dy}{\sqrt{(a_1, \ldots)(y, 1)^4}},$$

where if

$$I = ae - 4bd + 3c^2,$$

$$J = ace - ad^2 - b^2e + 2bcd - c^3,$$

and similarly I_1, J_1, are the invariants of the two functions, then Δ, Δ_1 are the discriminants

$$\Delta = I^3 - 27J^2, \quad \Delta_1 = I_1^3 - 27J_1^2,$$

and the γ_2, γ_3 of Kiepert's equations are

$$\gamma_2 = I \div \sqrt[3]{\Delta}, \quad \gamma_3 = J \div \sqrt{\Delta},$$

whence

$$\gamma_2^3 - 27\gamma_3^2 = 1.$$

In particular, if the forms are

$$1 - x^2 . 1 - k^2x^2, \text{ and } 1 - y^2 . 1 - \lambda^2 y^2,$$

and if as usual $k = u^4$, $\lambda = v^4$, and M is the multiplier for the form

$$\frac{dx}{\sqrt{1-x^2 . 1-k^2x^2}} = \frac{Mdy}{\sqrt{1-y^2 . 1-\lambda^2y^2}}$$

then we have

$$I = \tfrac{1}{12}(1 + 14u^8 + u^{16}),$$

$$J = \tfrac{1}{216}(1 + u^8)(1 - 34u^8 + u^{16}),$$

$$\Delta = \tfrac{1}{16}u^8(1-u^8)^4, \quad \Delta_1 = \tfrac{1}{16}v^8(1-v^8)^4,$$

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2}\,\frac{1 + 14u^8 + u^{16}}{u^{\frac{8}{3}}(1-u^8)^{\frac{4}{3}}}, \quad \gamma_3 = \tfrac{1}{54}\frac{(1+u^8)(1-34u^8+u^{16})}{u^4(1-u^8)^2},$$

and thence

$$L^2 = \frac{v^{\frac{2}{3}}(1-v^8)^{\frac{1}{3}}}{u^{\frac{2}{3}}(1-u^8)^{\frac{1}{3}}}\,\frac{1}{M},$$

which last equation is the expression for L^2 in terms of the Jacobian symbols u, v, M.

As an easy verification in a particular case, suppose $n = 5$. We have here

$$L^2 = \frac{v^{\frac{2}{3}}(1-v^8)^{\frac{1}{3}}}{u^{\frac{2}{3}}(1-u^8)^{\frac{1}{3}}} . \frac{1}{M}, \quad M = \frac{v(1-uv^3)}{v-u^5}, \quad \left(= \frac{v+u^5}{5u(1+u^3v)}\right),$$

$$u^6 - v^6 + 5u^2v^2(u^2 - v^2) + 4uv(1 - u^4v^4) = 0,$$

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2}\,\frac{1 + 14u^8 + u^{16}}{u^{\frac{8}{3}}(1-u^8)^{\frac{4}{3}}};$$

and it should be possible, by eliminating u, v, M, to deduce hence the L-equation

$$L^{12} + 10L^6 - 12\gamma_2 L^2 + 5 = 0. \quad \text{(Kiepert, p. 428.)}$$

It does not seem in any wise easy to do this in the case of an arbitrary modulus; but writing the modular equation in the form

$$(u^2 - v^2)(u^4 + 6u^2v^2 + v^4) + 4uv(1 - u^4v^4) = 0,$$

we satisfy this by

$$uv - 1 = 0, \quad u^4 + 6u^2v^2 + v^4 = 0,$$

or say by

$$v = \frac{1}{u}, \quad u^8 + 6u^4 + 1 = 0,$$

and the equation may be verified for this particular modulus.

We have

$$1 + 14u^8 + u^{16} = 48u^8, \quad (1-u^8)^2 = 32u^8,$$

and consequently

$$\gamma_2 = \tfrac{1}{6}\sqrt[3]{2} . \frac{48u^8}{u^{\frac{8}{3}}(32u^8)^{\frac{2}{3}}}, \quad = \frac{1}{2.3}\,2^{\frac{1}{3}}\,\frac{2^4 . 3u^8}{u^{\frac{8}{3}} . 2^{\frac{10}{3}} . u^{\frac{16}{3}}} = 1, \text{ (whence also } \gamma_3 = 0).$$

Moreover

$$M=\frac{\dfrac{1}{u}-\dfrac{u}{u^4}}{\dfrac{1}{u}-u^5}=-\frac{1}{u^2}\frac{1-u^2}{1-u^6}=\frac{-1}{u^2(1+u^2+u^4)},$$

and thence

$$L^2=\frac{\dfrac{1}{u^{\frac{2}{3}}}\left(1-\dfrac{1}{u^8}\right)^{\frac{1}{3}}}{u^{\frac{2}{3}}(1-u^8)^{\frac{1}{3}}}\frac{1}{M}=\frac{(u^8-1)^{\frac{1}{3}}}{u^4(1-u^8)^{\frac{1}{3}}}\frac{1}{M}=-\frac{1}{u^4M},$$

that is,

$$L^2=1+u^2+u^{-2}.$$

But we have

$$u^4=-3+2\sqrt{2},$$

and thence

$$u^2=\quad i(1-\sqrt{2}),\quad u^{-2}=i(1+\sqrt{2}),$$

and

$$L^2=\quad 1+2i,$$

whence

$$L^{12}+10L^6-12\gamma_2L^2+5=(117+44i)+10(-11-2i)-12(1+2i)+5=0,$$

or the L-equation is satisfied.

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