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ON MULTIPLE ALGEBRA.

1. I REPRODUCE a passage from my Presidential Address, British Association, Southport, 1883.

"Outside of ordinary mathematics we have some theories which must be referred to: algebraical, geometrical, logical. It is, as in many other cases, difficult to draw the line: we do in ordinary mathematics use symbols not denoting quantities, which we nevertheless combine in the way of addition and multiplication, $a + b$ and ab, and which may be such as not to obey the commutative law $ab = ba$; in particular, this is or may be so in regard to symbols of operation; and it could hardly be said that any development whatever of the theory of such symbols of operation did not belong to ordinary algebra. But I do separate from ordinary algebra the system of multiple algebra or linear associative algebra developed in the valuable memoir by the late Benjamin Peirce, "Linear Associative Algebra" (1870, reprinted 1881 in the *American Journal of Mathematics*, vol. IV., with notes and addenda by his son, C. S. Peirce): we here consider symbols A, B, &c., which are linear functions of a determinate number of letters or units i, j, k, l, &c., with coefficients which are ordinary analytical magnitudes real or imaginary, viz. the coefficients are in general of the form $x + iy$, where i is the before-mentioned imaginary, or $\sqrt{(-1)}$ of ordinary analysis. The letters i, j, k, &c., are such that every binary combination i^2, ij, ji, &c., (ij not in general $= ji$), is equal to a linear function of the letters, but under the restriction of satisfying the associative law; viz. for each combination of three letters $ij.k$ is $= i.jk$, so that there is a determinate and unique product of three or more letters; or, what is the same thing, the laws of combination of the units i, j, k, ... are defined by a multiplication table giving the values of i^2, ij, ji, &c.; the original units may be replaced by linear functions of these units, so as to give rise for the units finally adopted to a multiplication table of the most simple form; and it is very remarkable how

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frequently in these simplified forms we have nilpotent or idempotent symbols ($i^2 = 0$ or $i^2 = i$ as the case may be), and symbols i, j, such that $ij = ji = 0$; and consequently how simple are the forms of the multiplication tables which define the several systems respectively.

"I have spoken of this multiple algebra before referring to various geometrical theories of earlier date, because I consider it as the general analytical basis, and the true basis, of these theories. I do not realise to myself directly the notions of the addition or multiplication of two lines, areas, rotations, or other geometrical, kinematical, or mechanical entities; and I would formulate a general theory as follows: consider any such entity as determined by the proper number of parameters a, b, c, ... (for instance, in the case of a finite line given in magnitude and position, these might be the length, the coordinates of one end, and the direction-cosines of the line considered as drawn from this end); and represent it by or connect it with the linear function $ai + bj + ck + $ &c., formed with these parameters as coefficients and with a given set of units i, j, k, &c. Conversely, any such linear function represents an entity of the kind in question. Two given entities are represented by two linear functions; the sum of these is a like function representing an entity of the same kind, which may be regarded as the sum of the two entities; and the product of them (taken in a determined order, when the order is material) is an entity of the same kind, which may be regarded as the product (in the same order) of the two entities. We thus establish by definition the notion of the sum of the two entities, and that of the product (in a determinate order, when the order is material) of the two entities. The value of the theory in regard to any kind of entity would of course depend on the choice of a system of units i, j, k, ..., with such laws of combination as would give a geometrical or kinematical or mechanical significance to the notions of the sum and product as thus defined.

"Among the geometrical theories referred to, we have a theory (that of Argand, Warren, and Peacock) of imaginaries in plane geometry; Sir W. R. Hamilton's very valuable and important theory of quaternions; the theories developed in Grassmann's *Ausdehnungslehre*, 1844 and 1862; Clifford's theory of biquaternions, and recent extensions of Grassmann's theory to non-Euclidian space by Mr Homersham Cox. These different theories have of course been developed, not in anywise from the point of view from which I have been considering them, but from the points of view of their several authors respectively."

2. The present paper is in a great measure the development of the views contained in the foregoing extract; but, instead of establishing *ab initio* a linear function $ai + bj + ck + ...$ as above, I deduce this, as will be seen from the notion of addition.

3. If x, y, ... denote ordinary (real or imaginary) analytical magnitudes, which (as such) are susceptible of addition and multiplication, and for each of these operations are commutative and associative, then we may consider a multiple symbol $(x, y, ...)$ involving any given number of letters, to fix the ideas say (x, y), susceptible of addition and multiplication according to determinate laws

$$(x,\ y) + (x',\ y') = (P,\ Q), \quad (x,\ y)(x',\ y') = (X,\ Y),$$

where P, Q, X, Y are given functions of x, y, x', y'. For greater simplicity the law of addition is taken to be

$$(x, y) + (x', y') = (x + x', y + y'),$$

so that as regards addition the multiple symbols are commutative and associative. But this is or is not the case for multiplication, according to the form of the given functions X, Y; for instance, if

$$(x, y)(x', y') = (xx' - yy', xy' + yx'),$$

then in regard to multiplication the symbols will be commutative and associative. But if

$$(x, y, z)(x', y', z') = (yz' - y'z, zx' - z'x, xy' - x'y),$$

then the symbols will be associative, but not commutative.

4. I remark here that we are in general concerned with symbols of a given multiplicity, double symbols (x, y), triple symbols (x, y, z), n-tuple symbols $(x_1, x_2, ..., x_n)$, as the case may be, and that as well the product as the sum is a symbol *ejusdem generis*, and consequently of the same multiplicity, with the component symbols; this is to be assumed throughout in the absence of an express statement to the contrary. It is, moreover, proper to narrow the notion of multiplication by restricting it to the case where the terms $(X, Y, ...)$ of the product are linear functions of the terms $(x, y, ...)$ and $(x', y', ...)$ of the component symbols respectively; any other form $(X, Y, ...)$ is better designated *not as a product*, but as a combination (or by some other name) of the component symbols $(x, y, ...)$ and $(x', y', ...)$.

5. I assume, moreover, that, if m be any ordinary analytical magnitude, this may be multiplied into a multiple symbol $(x, y, ...)$, according to the law

$$m(x, y, ...) = (mx, my, ...).$$

6. As a consequence of this last assumption and of the assumed law of addition, we have for instance

$$(x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)$$
$$= x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1);$$

that is, using single letters i, j, k for the multiple symbols $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively, we have

$$(x, y, z) = xi + yj + zk,$$

where the letters i, j, k, thus standing for determinate multiple symbols, may be termed "extraordinaries." Each extraordinary may be multiplied into any ordinary symbol x, and is commutative therewith, $xi = ix$; moreover, each extraordinary may be multiplied into itself, or into another extraordinary, according to laws which are, in fact, determined by means of the assumed law of multiplication of the original multiple symbols; and, conversely, the law of multiplication of the extraordinaries determines that of the original multiple symbols; thus, if

$$(x, y)(x', y') = (xx' - yy', xy' + yx'),$$

then

$$(ix + jy)(ix' + jy') = i(xx' - yy') + j(xy' + yx'),$$

and also

$$= i^2 xx' + ij xy' + ji yx' + j^2 yy',$$

which expressions will agree together if, and only if, $i^2 = i$, $ij = j$, $ji = j$, $j^2 = -i$, or, as these equations may be written,

	i	j
i	i	j
j	j	$-i$

And so in general we have a multiplication table giving each square or product as a homogeneous linear function of all or any of the extraordinaries.

7. And, conversely, from this multiplication table of the extraordinaries i, j, we have

$$(ix + jy)(ix' + jy') = i(xx' - yy') + j(xy' + yx'),$$

that is,

$$(x, y)(x', y') = (xx' - yy', \ xy' + yx')$$

the originally assumed formula of multiplication.

In the example just given, we have $i^2 = i$, $ij = ji = j$, viz. the symbol i comports itself like unity, and may be put $= 1$; we have $(x, y) = x + jy$, with the multiplication table

	1	j
1	1	j
j	j	-1

or simply the equation $j^2 = -1$; and then

$$(x + jy)(x' + jy') = xx' - yy' + j(xy' + yx');$$

and it is convenient to regard 1 as an extraordinary, and speak of the system of extraordinaries 1, j.

8. The separate terms $x, y, \ldots$, whatever be their number, may always without loss of generality be arranged in a line; but it may be convenient to arrange them in a different form, for instance, in that of a square; we may have for instance symbols $\begin{vmatrix} x, & y \\ z, & w \end{vmatrix}$ with the laws of combination

$$\begin{vmatrix} x, & y \\ z, & w \end{vmatrix} + \begin{vmatrix} x', & y' \\ z', & w' \end{vmatrix} = \begin{vmatrix} x + x', & y + y' \\ z + z', & w + w' \end{vmatrix},$$

$$\begin{vmatrix} x, & y \\ z, & w \end{vmatrix} \cdot \begin{vmatrix} x', & y' \\ z', & w' \end{vmatrix} = \begin{vmatrix} xx' + yz', & xy' + yw' \\ zx' + wz', & zy' + ww' \end{vmatrix},$$

where observe that the multiplication is not commutative.

9. A multiple symbol may be connected with a geometrical or physical entity of any kind : viz. any such entity, depending on a number of parameters susceptible of analytical magnitude, to fix the ideas say on two parameters (x, y), may be connected with the multiple symbol (x, y). We cannot in general directly conceive the addition or multiplication of such entities, but the assumed laws of combination of the multiple symbols in effect serve as definitions of the operations. Thus

$$(x, y) + (x', y') = (x + x', y + y'); \quad (x, y) . (x', y') = (xx' - yy', xy' + yx'):$$

or the sum of the entities whose parameters are x, y and x', y' is by definition the entity whose parameters are $x + x'$, $y + y'$; and the product of the same entities is by definition the entity whose parameters are $xx' - yy'$, $xy' + yx'$. The entities are thus, as regards addition, commutative and associative; but, as regards multiplication they are or are not commutative, or associative, according to the assumed law of multiplication of the multiple symbols.

10. If, as above, the multiple symbol be represented by means of extraordinaries, $(x, y) = ix + jy$, then the extraordinaries i, j, quâ multiple symbols $(1, 0)$, $(0, 1)$, are themselves special entities of the kind in question, their laws of combination (as such special entities) being included in the general laws for the combination of such entities ; or, if we please, these general laws being derived from the assumed laws for the special entities. Thus, if $ix + jy$ represent the point whose coordinates are x, y, then i represents the point whose coordinates are $(1, 0)$, and j the point whose coordinates are $(0, 1)$.

11. It has been assumed that the multiple symbols combined together, and their sum and product, are symbols of one and the same multiplicity, say the types of combination are

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z'); \quad (x, y, z)(x', y', z') = (X, Y, Z);$$

if this were not so, if for instance we had

$$(x, y, z)(x', y', z') = \text{a double symbol } (X, Y),$$

then this given equation for the multiplication of two triple symbols would not serve as a definition for the multiplication of double symbols, or of a double and a triple symbol, and new definitions would be required. We might, however, have symbols of indefinite multiplicity $(x, y, z, w, ...)$, including within them all finite multiplicities, viz. (x, y) meaning $(x, y, 0, 0, ...)$, and so in other cases, these being combined into symbols of like indefinite multiplicity

$$(x, y, z, ...) + (x', y', z', ...) = (x + x', y + y', z + z', ...);$$
$$(x, y, z, ...)(x', y', z', ...) = (X, Y, Z, ...);$$

for instance, the law of multiplication might be

$$(x, y, z, ...)(x', y', z', ...) = (xx', xy' + yx', xz' + yy' + zx', ...),$$

and we could hereby combine symbols of any finite multiplicities (the same or different) whatever

12. Another peculiarity may be noticed: suppose that, in general,

$$(x,\ y,\ z,\ w) + (x',\ y',\ z',\ w') = (x + x',\ y + y',\ z + z',\ w + w');$$

$$(x,\ y,\ z,\ w)(x',\ y',\ z',\ w') = (X,\ Y,\ Z,\ W);$$

then, as a particular case hereof, we have the addition-equation

$$(x,\ y,\ 0,\ 0) + (x',\ y',\ 0,\ 0) = (x + x',\ y + y',\ 0,\ 0)$$

and it may very well be that for the same two symbols the multiplication-equation may take the form

$$(x,\ y,\ 0,\ 0)(x',\ y',\ 0,\ 0) = (0,\ 0,\ Z,\ W),$$

(Z, W each of them a function of x, y, x', y'); viz. here, in a different point of view, the component symbols $(x,\ y,\ 0,\ 0)$ and $(x',\ y',\ 0,\ 0)$, are double systems of a certain kind $\{x,\ y\}$, $\{x',\ y'\}$, and the product is a double system of a different kind $[Z,\ W]$, or it might have been $(0,\ Y,\ Z,\ W)$, = a triple system $[Y,\ Z,\ W]$. Of course, all peculiarity disappears when we revert from the particular symbols $(x,\ y,\ 0,\ 0)$ to the general symbols $(x,\ y,\ z,\ w)$, from which we may regard them as derived.

13. The peculiarity just referred to presents itself naturally when we regard the symbols as representing geometrical or physical entities; it may very well be that the product of two entities of a certain kind is taken to be an entity of a different kind (for instance, the product of two points to be a line), and that the analytical theory of the multiple symbol is constructed in order to such a relation; and it may further be that only the symbol $(x,\ y,\ 0,\ 0)$ or $\{x,\ y\}$ is interpreted with reference to an entity of the one kind, and only the symbol $(0,\ 0,\ Z,\ W)$, or $[Z,\ W]$ is interpreted by reference to an entity of the other kind, without any interpretation at all being given to the general symbol $(x,\ y,\ z,\ w)$; thus the two forms $(x,\ y,\ 0,\ 0)$, and $(0,\ 0,\ Z,\ W)$, naturally present themselves as double symbols $\{x,\ y\}$ and $[Z,\ W]$ of different kinds.

14. In further illustration, suppose the symbols represented as linear functions of extraordinaries,

$$(x,\ y,\ z,\ w) = x\epsilon_1 + y\epsilon_2 + z\eta_1 + w\eta_2,$$

with a multiplication table for the four extraordinaries ϵ_1, ϵ_2, η_1, η_2. We then have $(x,\ y,\ 0,\ 0)$, $= \{x,\ y\} = x\epsilon_1 + y\epsilon_2$; the ϵ_1, ϵ_2 have no proper multiplication table of their own, but the squares and products ϵ_1^2, $\epsilon_1\epsilon_2$, $\epsilon_2\epsilon_1$, ϵ_2^2 are each given as a linear function of η_1, η_2; hence we have

$$(x\epsilon_1 + y\epsilon_2)(x'\epsilon_1 + y'\epsilon_2) = Z\eta_1 + W\eta_2,$$

which is a symbol $(0,\ 0,\ Z,\ W)$, or $[Z,\ W]$, of a different kind; it may, in completion of the theory, be assumed that in like manner η_1, η_2 have no proper multiplication table of their own, but that the squares and products η_1^2, $\eta_1\eta_2$, $\eta_2\eta_1$, η_2^2 are each given as a linear function of ϵ_1, ϵ_2, leading to a relation

$$(z\eta_1 + w\eta_2)(z'\eta_1 + w'\eta_2) = X\epsilon_1 + Y\epsilon_2,$$

which is a symbol $(X, Y, 0, 0)$ or $\{X, Y\}$ of the first kind; it may further happen that each of the several products $\epsilon\eta$, $\eta\epsilon$ is $= 0$, in which case

$$(x\epsilon_1 + y\epsilon_2)(z\eta_1 + w\eta_2) = 0, \quad (z\eta_1 + w\eta_2)(x\epsilon_1 + y\epsilon_2) = 0.$$

For the complete theory, we require the multiplication table of the four extraordinaries ϵ_1, ϵ_2, η_1, η_2. The geometrical interpretation may be that $x\epsilon_1 + y\epsilon_2$ represents a point, $z\eta_1 + w\eta_2$ a line (with or without any geometrical interpretation of a symbol $x\epsilon_1 + y\epsilon_2 + z\eta_1 + w\eta_2$), the product of two points is a line, the product of two lines is a point; that of a line and point is $= 0$, (see *post* Grassmann, where however the system is a somewhat different one).

15. I do not propose in the present paper to consider the subject of multiple algebra from an analytical point of view; and I will make only a few remarks and give some references. The general theory of associative linear forms is treated in a very satisfactory manner in Peirce's Memoir (1870) above referred to, only it is assumed throughout *ab initio* that the forms are associative. In a Note "On Associative Imaginaries," *Johns Hopkins Circulars*, No. 15 (1882), [822], and more fully in the paper "On Double Algebra," *Proc. Lond. Math. Soc.*, t. XV. (1884), pp. 185—197, [814], starting from the assumed equations $x^2 = ax + by$, $xy = cx + dy$, $yx = ex + fy$, $y^2 = gx + hy$, between the extraordinaries x, y, I considered under what conditions the algebra of these symbols was in fact associative. And in a paper "On the 8-square Imaginaries," *American Journal of Mathematics*, t. IV. (1881), pp. 293—296, [773], I showed that the extraordinaries 0, 1, 2, 3, 4, 5, 6, 7, connected with Euler's theorem of the 8 squares, are of necessity *non-associative*. I do not know that anything else has been done in regard to non-associative algebras: and it thus appears that the question has hardly been discussed at all. Matrices are associative: I shall have again to refer to them.

16. The object of the present paper is to discuss in detail, from the point of view explained in the extract from my British Association Address, the different theories in regard to geometrical or other entities. I do this under various headings.

The $i = \sqrt{(-1)}$ of Analysis and Analytical Geometry.   Art. Nos. 17—26.

17. We may, of course, consider the i as an extraordinary. We have a double symbol (x, y), combining according to the laws

$$(x, y) + (x', y') = (x + x', y + y'),$$
$$(x, y)(x', y') = (xx' - yy', xy' + yx');$$

and then, introducing the extraordinaries k, i, and writing

$$(x, y) = kx + iy,$$

we have

$$(kx + iy)(kx' + iy') = k^2xx' + kixy' + ikyx' + i^2yy';$$

whence

$$k^2 = k, \quad ki = ik = i, \quad i^2 = -k,$$

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or the multiplication table is

	k	i
k	k	i
i	i	$-k$

.

But the conditions are satisfied by, and we accordingly assume, $k = 1$; and there then remains only the condition $i^2 = -1$. Compare herewith Sir W. R. Hamilton, "Theory of Conjugate Functions, or Algebraical Couples; with a Preliminary and Elementary Essay on Algebra as a Science of pure time" (*Trans. R. I. Acad.*, t. XVII., 1833—35). I refer to this paper in my Address, and remark upon it that I cannot appreciate the manner in which the author connects, with the notion of time, his algebraical couple or imaginary magnitude $a + bi$, $= a + b \sqrt{(-1)}$ as written in the memoir.

18. But we have, in fact, passed out of this view, and have come to regard $a + bi$ as an ordinary analytical magnitude; viz. in every case an ordinary symbol represents or may represent such a magnitude, and the magnitude (and as a particular case thereof, the symbol i) is commutable with the extraordinaries of any system of multiple algebra. And similarly in analytical geometry, without seeking for any real represent-ation, we deal with imaginary points, lines, &c., that is, with points, lines, &c., depending on parameters of the foregoing form $a + bi$.

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$\sqrt{(-1)}$ *denotes Perpendicularity.* Art. Nos. 19—26.

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19. I give a list of the earlier works and memoirs, marking with an asterisk those which I have not seen.

Wallis. *De Algebrâ Tractatus*, 1685 Anglice editus; 1693. Chapters 66, 67, 68, and 69 have respectively the titles: De quadratis negativis eorumque radicibus dictis Imaginariis;—Eorundem exemplificatio in Geometriâ;—Effectiones geometricæ his accom-modatæ; Aliæ quæ huc spectant constructiones geometricæ. I quote a single sentence from Chap. 67: "Prout igitur cum æquationis quadraticæ radix prodit negativa, dicendum verbi gratiâ punctum B pro eo statu haberi non posse ut supponitur in expositâ AC prorsum; posse tamen retrorsum ab A in eâdem rectâ—hic vero (de radice quadrati negativi) dicendum, non haberi quidem posse punctum B ut erat suppositum in AC rectâ, vel ante vel retro: posse tamen (in eodem plano) suprà rectam illam": viz. the distance represented by the square root of a negative quantity cannot be measured in the line, forwards or backwards; but can be measured (in the same plane) above the line—or, as appears elsewhere, at right angles to the line, either in the plane, or in a plane at right angles thereto.

Forcenex. "Réflexions sur les quantités imaginaires," *Mel. de Turin*, t. I. (1759), pp. 113—146. The author, after stating that there was no geometrical construction

for imaginary quantities, proceeds "Cependant, pour conserver une certaine analogie avec les quantités négatives, un auteur dont nous avons un cours d'algèbre d'ailleurs fort estimable a prétendu les devoir prendre sur une ligne perpendiculaire à celle où l'on les avait supposées, si par exemple &c.," the example which he considers being as follows: On a given line AB to find a point P, such that the product of the distances AP, BP is $=\frac{1}{2}(AB)^2$. Taking $AB = 2a$ and $AP = x$, then $x(2a - x) = 2a^2$, that is, $(x - a)^2 = -a^2$, or, $x = a + a\sqrt{(-1)}$, an imaginary value; the condition is, however, satisfied by a real point P *off* the line AB

$$(AM = MP = \tfrac{1}{2}AB = a), \text{ (fig. 1)},$$

and hence the author in question infers that the imaginary value $a\sqrt{(-1)}$ is represented by the line MP at right angles to AB. Forcenex objects to this: if the

Fig. 1.
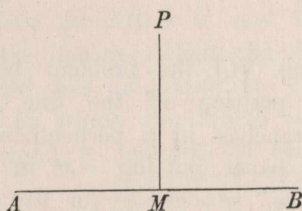
point P may be taken off the axis of x, then the condition $AP \cdot BP = 2a^2$ gives for the locus of P a certain curve; and there is no reason why the foregoing solution $x = a + a\sqrt{(-1)}$ should represent one point rather than another of this curve. As to this see *post* (Peacock).

*__Truel,__ Henri Dominique, 1786.

*__Suremain-de-Missery.__ *Théorie purement algébrique des quantités imaginaires,* Paris, 1801; it is referred to in the letter by Servois, *infra.*

20. __Buée.__ "Mémoire sur les quantités imaginaires," *Phil. Trans.,* 1806, pp. 23—88.

I give an extract: "Du signe $\sqrt{(-1)}$. Je mets en titre, du signe $\sqrt{(-1)}$ et non de la quantité imaginaire ou de l'unité imaginaire, parce que $\sqrt{(-1)}$ est un signe particulier joint à l'unité réelle, non une quantité particulière: c'est un nouvel adjectif joint au substantif ordinaire et non un nouveau substantif.

"Mais que veut dire ce signe? il n'indique ni une addition ni une soustraction, ni une suppression ni une opposition par rapport aux signes + et −: une quantité accompagnée par $\sqrt{(-1)}$ n'est ni additive ni subtractive, ni égale à zéro. La qualité marquée par $\sqrt{(-1)}$ n'est opposée ni à celle qu'indique +, ni à celle marquée par −; qu'est-elle donc?

"Pour la découvrir supposons trois lignes égales AB, AC, AD (fig. 2) qui partent toutes du point A. Si je désigne la ligne AB par $+1$, la ligne AC sera -1, et la ligne AD qui est une moyenne proportionnelle entre $+1$ et -1, sera nécessairement $\sqrt{(-1^2)}$ ou plus simplement $\sqrt{(-1)}$. Ainsi $\sqrt{(-1)}$ est le signe de la PERPENDICULARITÉ: donc la propriété caractéristique est *que tous les points de la perpendiculaire sont*

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également éloignés de points placés à égale distance de part et d'autre de son pied. Le signe $\sqrt{(-1)}$ signifie tout cela et il est le seul qui l'exprime.

Fig. 2.
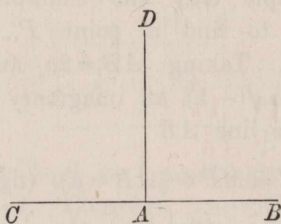
"*Ce signe mis devant a (a signifiant une ligne ou une surface) veut donc dire qu'il faut donner à a une situation perpendiculaire à celle qu'on lui donnerait si l'on avait simplement a ou − a.*"

And he, in fact, gives (Prob. VI.) the problem before referred to, and finds no difficulty in assigning to P a position off the line AB. I notice also that Buée considers a curve as having branches in a perpendicular plane; thus for the circle $x^2 + y^2 = a^2$ in the plane of the paper, putting $-x^2$ in place of x^2, there are branches, or, as he calls it, "une appendice," $y^2 = a^2 + x^2$ (or say $y^2 = a^2 + z^2$), being a rectangular hyperbola, in a plane at right angles to the plane of the paper.

21. Other writings are:

Argand. *Essai sur une manière de représenter les quantités imaginaires.* Paris, 1806. Partly reproduced in Argand's memoir under the same title presently referred to.

Francais, J. F. "Nouveaux principes de géométrie de position et interprétation géométrique des symboles imaginaires." *Gergonne Annales*, t. IV. 1813—14, pp. 61—72.

Argand. "Essai sur une manière, &c." *Ibid.* pp. 133—148.

Francais, J. F. "Lettre au Rédacteur." *Ibid.* pp. 222—227.

Servois. "Lettre au Rédacteur." *Ibid.* pp. 228—235.

This last is *against* the theory, with a running commentary of notes *in favour* by Gergonne.

Francais, J. F. "Autre lettre." *Ibid.* pp. 364—366.

Lacroix. "Note transmise à M. Vecten"; it calls attention to Buée's memoir. *Ibid.* p. 367.

*Mourey. *Vraie théorie des quantités négatives et des quantités prétendues imaginaires*, 1828, reprinted 1861.

22. **Warren, J.** *A treatise on the geometrical representation of the square roots of negative quantities.* 8vo. Cambridge, 1828, pp. 1—154.

Peacock, Preface to *Algebra*, 1830, speaks of Warren's work as "distinguished for great originality and for extreme boldness in the use of definitions." There is no Preface or Introduction; the first Chapter is entitled Definitions, Addition, Subtraction, Proportion, Multiplication, Division, Fractions, and Raising of Powers. I quote certain articles as follows:

(1) All straight lines drawn in a given direction from a given point are represented in *length* and *direction* by algebraic quantities; and in the following treatise whenever the word *quantity* is used it is to be understood as signifying *a line.*

(2) DEF. The given point from which the straight lines are measured is called the *origin.*

(3) DEF. The *sum* of two quantities is the diagonal of the parallelogram whose sides are the two quantities.

(12) DEF. The first of four quantities is said to have to the second the same ratio which the third has to the fourth: when the first has in *length* to the second the same ratio which the third has in *length* to the fourth, according to Euclid's definition; and also the angle at which the fourth is inclined to the third is equal to the angle at which the second is inclined to the first, and is measured in the same direction.

(17) DEF. Unity is a positive quantity arbitrarily assumed, from a comparison of which the values of other quantities are obtained.

(18) DEF. If there be three quantities such that unity is to the first as the second is to the third, then the third is called the *product* which arises from the *multiplication* of the second by the first.

23. The signification of these definitions may be thus expressed. Let the line, drawn from an origin O to the point whose rectangular coordinates are x, y, be called the line $x + iy$; of course the line, length unity in the direction of the axis of x, is the line 1. Then, observing that, putting x, $y = r \cos \theta$, $r \sin \theta$, and x', $y' = r' \cos \theta'$, $r' \sin \theta'$, we have $(x + iy)(x' + iy') = rr' \{\cos (\theta + \theta') + i \sin (\theta + \theta')\}$, the two definitions are:

The *sum* of the lines $x + iy$ and $x' + iy'$ is the line

$$x + x' + i(y + y');$$

and the *product* of the same lines is the line

$$(x + iy)(x' + iy');$$

viz. Warren in effect establishes by definition the notions of the sum and product of two lines, and that by connecting the line with a linear symbol $x + iy$.

It may be right to remark that the general notion of a proportion, used as above to define multiplication, is geometrically the more simple of the two, and that the condition for the proportion $OA : OB = OC : OD$ may be thus expressed: the lengths of the lines are proportional, and the inclination OA to OB is equal to the inclination OC to OD, these inclinations being in the same sense.

24. **Peacock, G.** *A Treatise on Algebra.* 8vo. Cambridge, 1830 (one Vol., pp. 5—38 and 1—685).

——— "Report on the Recent Progress and Present State of Certain Branches of Analysis." *B. A. Report,* (1833), pp. 185—352.

——— *A Treatise on Algebra.* Vol. I. Arithmetical Algebra; Vol. II. On Symbolical Algebra and its Application to the Geometry of Position. 8vo. Cambridge, 1842 and 1845.

The statement of the theory is substantially identical with that of the earlier writers, thus *Algebra,* (1830), p. 362, No. 439, is:—"We have shown on a former occasion that, if a designated a line in one direction, $-a$ must designate a line in the direction opposite to it, or making an angle with the former equal to two right angles or 180 degrees; but inasmuch as $a\{\surd(-1)\}^2 = -a$, it follows that the *double affectation* of the line a with the sign $\surd(-1)$ produces a result represented by $-a$, and is consequently equivalent to its transfer through 180°: it follows therefore that its single affectation with the sign $\surd(-1)$ is equivalent to its transfer through half that angle, or through 90°; or, in other words, if a represents a line, $a\surd(-1)$ will represent a line at right angles to it." But further, pp. 666, 667, the Author considers the before-mentioned problem (for convenience, I write $2a$ instead of a) in the form "to divide a line $2a$ into two such parts that the rectangle contained by them may be $= b^2$"; he takes x, y for the two parts, and in the case where $b^2 > a^2$, finds the two parts $x = a \pm \surd(-1)\surd(b^2 - a^2)$, $y = a \mp \surd(-1)\surd(b^2 - a^2)$, which he interprets as meaning the two parts are AC, CB, where C is a point off the line AB.

25. It seems to me that, except in Warren, the defect in the exposition of the theory is that it is not made clear, that the theory is in fact a theory of lines in a plane through a given point, with assumed definitions for the addition and for the multiplication of lines; thus in Peacock's problem "to divide a line a into two such parts that the product of them is $= b^2$," the real meaning is "given a line OA, $= a$, and a line OB, $= b^2$ (a and b^2 real and positive, so that each of these lines is in the given direction Ox), to find lines OP, OQ such that their sum may be equal to the given line OA, and their product equal to the given line OB." To solve it, take the two lines to be $x + iy = z$, and $x' + iy'$, $= z'$; then, working with z and z', we have $z + z' = 2a$, $zz' = b^2$, giving a solution which may be written in the two forms

$$\{z = a + \surd(a^2 - b^2),\quad z' = a - \surd(a^2 - b^2)\},$$
and
$$\{z = a + i\surd(b^2 - a^2),\quad z' = a - i\surd(b^2 - a^2)\},$$

viz. for $a > b$, the two lines lie each of them in the line Ox, while for $b > a$, they are inclined at equal angles on opposite sides of the line Ox. Or if we work with the real values x, y, x', y', then the same two equations give $x + x' = 2a$, $y + y' = 0$, $xx' - yy' = b^2$, $xy' + yx' = 0$; viz. writing $y' = -y$, then we have $x + x' = 2a$, $xx' + y^2 = b^2$, $y(x' - x) = 0$, whence either $x = x'$ or else $y = 0$, and we have the same two solutions as before. Observe that the second condition is $zz' = b^2$, not (as Forcenex understood it) the product of the lengths OP, $OQ = b^2$; and his objection is thus answered. But I

cannot but consider that the form of enunciation, "to divide the given line into two parts, such that their product shall be $= b^2$," does not express the meaning with sufficient clearness.

26. I have referred to Buée's notion of a curve as having branches in a perpendicular plane—this notion is generalised and developed in the papers:—

Gregory, D. F. "On the existence of branches of curves in several planes." *Camb. Math. Journ.*, t. I. (1839), pp. 259—266 (2nd Ed., pp. 284—292).

Walton. "On the general interpretation of equations between two variables in analytical geometry." *Do.* t. II. (1840), pp. 103—113.

—— "On the general theory of multiple points." *Do.* pp. 155—167.

And see also:—

Gregory's *Examples of the Processes of the Differential and Integral Calculus.* 8vo. Cambridge, 1841, pp. 172 et seq.;

and the criticism on these papers:—

Salmon. *A Treatise on the Higher Plane Curves.* 8vo. Dublin, 1852, The Note on Imaginary Points and Curves, pp. 301—306.

Geometrical representation of $x + iy$: Gauss. Art. Nos. 27—29.

27. The theory is that established by **Gauss** in the paper "Theoria Residuorum biquadraticorum Commentatio Secunda," *Comm. Gott. Recent.*, t. VII. (1832); *Werke*, t. II. pp. 95—148; viz. Gauss remarks that, in the same way that a real quantity x may be represented by means of a point on a line, so an imaginary (or complex) quantity $x + iy$ may be represented by means of a point the abscissa of which is $= x$, and its ordinate is $= y$. Taking m, m' as given complex quantities, he speaks of the points answering to the complex quantities mm', m, m', 1 as forming a *proportion* (thus in effect defining the product of two points M, M'). But he defers to another occasion all further discussion of the theory.

28. The direction in which the theory has been developed is as follows: Considering two complex values z, $= x + iy$, and Z, $= X + iY$, connected by an equation $\phi(Z, z) = 0$, then we have z represented as above by means of a point in a plane, and Z represented in like manner by means of another point in a different plane; to any given value of z or Z there corresponds a determinate number of values of Z or z; that is, to any given real point of either plane, there corresponds a determinate number of real points in the other plane: the equation thus establishes a correspondence between the points of the two planes; and it is known that this is an orthomorphic correspondence, viz. to any indefinitely small area of the one plane, there correspond in the other plane indefinitely small areas each of them of the same shape as the area in the first plane. We have thus a solution of the geometrical problem of orthomorphic projection; and in the analytical point of view, the theory is of great importance as exhibiting the relation to each other of complex variables connected by an equation.

29. It is to be remarked that, although the correspondence is primarily a correspondence between real points of the two planes respectively, yet in the ulterior development of the theory of this correspondence it may become necessary to consider imaginary points in the two planes. In view hereto, there would be some propriety in replacing the i of the z, Z by a like symbol I; say we have $z = x + Iy$, $Z = X + IY$ (where the x, y, X, Y may now be ordinary complex magnitudes $u + vi$; $i^2 = -1$ as before: $I^2 = -1$, and $Ii = iI$, viz. I is a square root of -1, commutable with the ordinary imaginary i). But this in passing.

The Barycentric Calculus; Möbius.   Art. Nos. 30, 31.

30. **Möbius,** *Der barycentrische Calcul*, 8vo. Leipzig, 1827, reprinted in Vol. I. of the *Gesammelte Werke*, Leipzig, 1885.

The theory applies to points in a line, in a plane, or in space; but it will be sufficient to consider the case of points in a plane. The idea is that, taking in the plane any three points A, B, C as fundamental points, then regarding these as loaded with the proper positive or negative weights p, q, r, any other point P of the plane may be regarded as the centre of gravity of these weights; say we have $P = pA + qB + rC$, as a representation of the point P; p, q, r are, in the first instance, given positive or negative real values (but they may be taken to be imaginary values), P, A, B, C are symbols denoting points. Observe that p, q, r are, in fact, trilinear coordinates: if for a moment ξ, η, ζ are the perpendicular distances of P from the sides of the triangle, and α, β, γ the perpendicular distances of the opposite vertices from the same sides respectively, then p, q, r are proportional to ξ/α, η/β, ζ/γ, or, what is the same thing, they are proportional to PBC/Δ, PCA/Δ, PAB/Δ, where PBC, &c., denote the three triangles and Δ the triangle ABC. It would be allowable to take p, q, r *equal* to these values, which would give $p + q + r = 1$. Möbius does this very nearly, for he writes $p + q + r + s = 0$, and then writing $-(p + q + r)D$ instead of P he has

$$pA + qB + rC + sD = 0;$$

that is,

$$D = -\frac{p}{s}A - \frac{q}{s}B - \frac{r}{s}C,$$

where

$$-\frac{p}{s} - \frac{q}{s} - \frac{r}{s} = 1:$$

writing x, y, z, instead of these quantities, I take therefore as the representation of the point, $P = xA + yB + zC$, where $x + y + z = 1$.

31. The symbols A, B, C may be regarded as extraordinaries; but it can hardly be said that Möbius so regards them, for he does not establish the notion of the multiplication of points, nor consequently any multiplication table for these symbols A, B, C: he does however deal with the *addition* of points, viz. if

$$P = xA + yB + zC, \quad P' = x'A + y'B + z'C,$$

then

$$P + \lambda P', \ = (x + \lambda x')\,A + (y + \lambda y')\,B + (z + \lambda z')\,C,$$

(where λ is indeterminate) denotes any point whatever in the line PP', (so also $P = xA + yB + zC$, where x, y, z are regarded as functions of a variable parameter θ, denotes any point whatever in the curve denoted by writing x, y, z proportional to such functions of θ). And he also considers (not quite generally) the question of the transformation of the fundamental points: this is really a treatment of the A, B, C as extraordinaries, the complete theory being as follows; assuming

$$A = \alpha A_1 + \beta B_1 + \gamma C_1, \quad B = \alpha' A_1 + \beta' B_1 + \gamma' C_1, \quad C = \alpha'' A_1 + \beta'' B_1 + \gamma'' C_1,$$

(if $\alpha + \beta + \gamma = 1$, $\alpha' + \beta' + \gamma' = 1$, $\alpha'' + \beta'' + \gamma'' = 1$, then these equations give conversely

$$A_1 = aA + a'B + a''C, \quad B_1 = bA + b'B + b''C, \quad C_1 = cA + c'B + c''C,$$

where $a + a' + a'' = 1$, $b + b' + b'' = 1$, $c + c' + c'' = 1$), then we have

$$P_1 = x\alpha + yB + zC_1 = (x\alpha + y\alpha' + z\alpha'') A_1 + (x\beta + y\beta' + z\beta'') B_1 + (x\gamma + y\gamma' + z\gamma'') C_1,$$

where the sum of the coefficients is

$$x(\alpha + \beta + \gamma) + y(\alpha' + \beta' + \gamma') + z(\alpha'' + \beta'' + \gamma''),$$

which is $= x + y + z$, $= 1$.

Equipollences: Bellavitis.   Art. Nos. 32, 33.

32. There are earlier papers in the years 1832 and 1833, but the method is explained in the Memoir:—

Bellavitis. "Saggio di applicazioni di un nuovo metodo di geometria analitica (Calcolo delle Equipollenze)," *Ann. Lomb. Veneto,* t. v. (1835), pp. 244—259,

and more fully in Memoirs, t. VII. (1837), and t. VIII. (1838); the author notices that his results were obtained independently of those of Möbius but that, on studying the *Barycentrische Calcul,* he had recognised that the two methods started from the same principles, and could easily be reduced to identity, but that they speedily separated from each other as well in regard to object as to form. But, in fact, the principles are rather those of Warren than of Möbius. They may be thus stated: 1°. two lines, equal, parallel and in the same sense, are said to be *Equipollent,* $AB \equiv CD$ (where the sign $\equiv$ is used instead of a special sign employed by Bellavitis); this first assumption in effect reduces the whole theory to that of lines drawn through a fixed point; 2°. $AB + BC \equiv AC$ (addition); and 3°. $AB \equiv \dfrac{CD \cdot EF}{GH}$, means not only that the lengths are connected by this relation, but that the inclination of AB to any fixed axis is $=$ inc. $CD +$ inc. $EF -$ inc. GH (proportion); or in particular, if GH be a line, length unity, in the direction of the axis, then $AB \equiv CD \cdot EF$, viz. the length $AB = $ product of lengths CD and EF; and, further,

$$\text{inc. } AB = \text{inc. } CD + \text{inc. } EF \text{ (multiplication)};$$

the agreement with Warren is thus complete.

33. The method is applied very elegantly to the solution of problems, for instance, t. v. p. 248; given two lines AB and CD, to find a point H, such that the triangles ABH, CDH may be directly similar, that is, such that they can be, by a rotation of one of them, brought to be similarly situate. The condition for this (a figure is easily supplied) is

$$\frac{AH}{AB} \equiv \frac{CH}{CD}, \text{ that is, } AH \cdot CD \equiv AB \cdot CH, \; = AB(AH - AC),$$

whence $AB \cdot AC = AH(AB - CD)$: or constructing a point E such that $BE \equiv DC$, viz. the line BE is drawn from B equal and parallel to and in the same sense with DC, then $AB - CD \equiv AE$, and the foregoing equation becomes

$$AH \cdot AE \equiv AB \cdot AC, \text{ or say } \frac{AH}{AC} \equiv \frac{AB}{AE},$$

viz. the required point H is such that the triangle ACH is directly similar to AEB. A solution of the like problem with the triangles ABH, CDH inversely similar is given, t. VIII., p. 27. I notice an expression t. VIII., p. 19, for the area of a triangle ABC,

$$= \frac{1}{4i}(BC \cdot \text{cj.} \, AB - AB \cdot \text{cj.} \, BC),$$

where cj. AB, conjugate of AB, is an equal line through A with an inclination $= -$ incl. AB; also a proof, t. VIII., p. 86, of the general theorem that an algebraical equation of any order has a root (this is, in a geometrical form, equivalent to the proofs given by Gauss and Cauchy); also t. VIII., p. 111, the notion of the " punti fittizj-conjugati," or real antipoints of a pair of conjugate imaginary points. The section " Delle figure a tre dimensioni," t. VIII., pp. 115—121, is confessedly quite incomplete, and without any anticipation of the quaternion or other three-dimensional imaginaries.

We have subsequently by the author

" Sposizione del Metodo delle Equipollenze," *Mem. Soc. Ital.*, t. XXV. (1855), pp. 225—309 : published separately, Modena, 1854 ;

and in French under the title

Exposition de la Méthode des Equipollences, par G. Bellavitis, traduit de l'italien par C. A. Laisant : 8vo. Paris, 1874.

Quaternions : Sir W. R. Hamilton. Art. Nos. 34—42.

34. I give the following references to early papers and systematic works :

Sir W. R. Hamilton. " Abstract of a paper On Quaternions or on a new system of imaginaries in Algebra, with some geometrical illustrations." *Proc. R. I. Acad.*, vol. III., Nov. 11, 1844, pp. 1—16. A note added during the printing refers to Cayley *infra*.

—— " On Quaternions ; or a new system of imaginaries in Algebra." *Phil. Mag.*, vol. XXV. (1844), pp. 489—495.

Cayley. " On certain results relating to Quaternions." *Phil. Mag.*, vol. XXVI. (1845), pp. 141—144, [20].

Sir W. R. Hamilton. *Lectures on Quaternions.* 8vo. Dublin, 1853.

—— *Elements of Quaternions.* 8vo. Dublin, 1866.

Tait, P. G. *An Elementary Treatise on Quaternions.* 8vo. Oxford, 1859.

Kelland, P. and **Tait, P. G.** *Introduction to Quaternions.* 8vo. London, 1873.

35. A quaternion is the sum of a scalar or mere magnitude w, plus a vector $ix + jy + kz$, which represents a line drawn from an origin O to the point whose rectangular coordinates are x, y, z; we have thus a quaternion Q, $= w + ix + jy + kz$, representing a line, and in connexion therewith a scalar w. Hamilton refers to the theory of Warren and Peacock as having in part suggested his investigations, but the contrast is very striking ; neither the form $x + iy$, where the directed magnitude x is unaccompanied by an extraordinary, nor the form $kx + iy$ with any multiplication table for k, i, is in any wise analogous to the expression $ix + jy + kz$ of a vector. We have, for addition, the ordinary formula $Q + Q' = w + w' + i(x + x') + j(y + y') + k(z + z')$; and then we have the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

or, what is the same thing, we have

$$i^2 = j^2 = k^2 = -1, \quad i = jk = -kj, \quad j = ki = -ik, \quad k = ij = -ji,$$

and thence for the product of two quaternions

$$(w + ix + jy + kz)(w' + ix' + jy' + kz')$$

$$= \quad ww' - xx' - yy' - zz'$$

$$+ i\,(wx' + xw' + yz' - y'z)$$

$$+ j\,(wy' + yw' + zx' - z'x)$$

$$+ k\,(wz' + zw' + xy' - x'y);$$

these, of course, include the forms for the addition and multiplication of two vectors $ix + jy + kz$ and $ix' + jy' + kz'$.

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But for the multiplication of two vectors, the form is simplified: viz. we have

$$VV' = (ix + jy + kz)(ix' + jy' + kz')$$
$$= -(xx' + yy' + zz') + i(yz' - y'z) + j(zx' - z'x) + k(xy' - x'y).$$

36. We may consider unit-vectors,

$$U = ix + jy + kz, \quad \text{where} \quad x^2 + y^2 + z^2 = 1,$$

and unit-quaternions

$$O = w + ix + jy + kz, \quad \text{where} \quad w^2 + x^2 + y^2 + z^2 = 1;$$

obviously the general form of a vector is $V = rU$, or, say the length is r, and the cosine-inclinations are x, y, z; similarly, the general form of a unit-quaternion is

$$O = \cos \delta + \sin \delta . U,$$

and that of a quaternion is

$$Q = TO = T(\cos \delta + \sin \delta . U),$$

T is the tensor, δ the amplitude, and x, y, z the cosine-inclinations of the unit-vector.

For the product of two unit-vectors, we have

$$UU' = -(xx' + yy' + zz') + i(yz' - y'z) + j(zx' - z'x) + k(xy' - x'y),$$

which is a unit-quaternion. Let δ, considered as a positive angle less than π, so that

$$\cos \delta = \pm , \quad \sin \delta = +,$$

be the inclination of the two vectors; we have $\cos \delta = xx' + yy' + zz'$. We can at right angles to the plane of U, U' draw in either of two opposite senses a unit-vector U''; and we may choose the sense in such wise that the rotation from U through angle δ to U' about U'' shall be in the sense Ox through angle $\frac{1}{2}\pi$ to Oy about Oz, say each of these is a right-handed rotation. This being so, we may write

$$i(yz' - y'z) + j(zx' - z'x) + k(xy' - x'y),$$
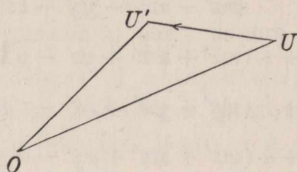$$= \sin \delta (ix'' + jy'' + kz'') = \sin \delta . U'',$$

and we have

$$UU' = -\cos \delta + \sin \delta . U'';$$

viz. the product is a unit-quaternion, the scalar part $= -\cos \delta$, and the vector part having a length $= \sin \delta$, and being at right angles to the plane of the two vectors U, U'; δ being, as above, the inclination of the two vectors. And similarly

$$U'U = -\cos \delta - \sin \delta . U''.$$

Fig. 3.

Representing the vectors as lines OU, OU', the product UU' will be represented, as in fig. 3, by means of an arrow drawn in the angle δ from U to U'; and of

course, for the product $U'U$, there would be the same figure with only the arrow drawn in the opposite sense from U' to U. The figure serves also to represent the unit-quaternion $-\cos\delta + \sin\delta \cdot U''$, which is the product of the two vectors. Observe that, in thus expressing a quaternion as a product of two vectors, only the plane of the vectors and the angle between them are determinate; we may rotate the angle UOU' in its own plane at pleasure.

In particular, if the two vectors are at right angles, then $\delta = 90°$, and the product is $UU' = U''$, viz. the product of the two unit-vectors at right angles to each other is a unit-vector at right angles to the plane of the two unit-vectors. If $\delta = 0$, then the two unit-vectors coincide, and we have $U^2 = -1$, viz. the square of a unit-vector is the scalar -1.

The product of any two vectors is given by the like formulæ; if V, $V' = rU$, $r'U'$ respectively, then $VV' = rr' \cdot UU'$.

37.   The product of a quaternion by a vector is given by the formula

$$QV' = (w + ix + jy + kz)(ix' + jy' + kz'),$$

$$= -(xx' + yy' + zz') + i(wx' + yz' - y'z) + j(wy' + zx' - z'x) + k(wz' + xy' - x'y).$$

In particular, if the quaternion and the vector are at right angles (that is, if the vector part of Q and V' are at right angles), $xx' + yy' + zz' = 0$, and the product is a vector. We may use this formula to express a given quaternion as the product of two vectors. Writing, for convenience,

$$U = i\xi + j\eta + k\zeta,$$

a unit-vector at right angles to Q, so that $\xi^2 + \eta^2 + \zeta^2 = 1$ and $\xi x + \eta y + \zeta z = 0$, then

$$QU = (w + ix + jy + kz)(i\xi + j\eta + k\zeta)$$
$$= i(w\xi + y\zeta - z\eta) + j(w\eta + z\xi - x\zeta) + k(w\zeta + x\eta - y\xi),$$

which is a vector, say it is

$$V = iX + jY + kZ;$$

and then

$$Q = -QU^2 = -VU = -(iX + jY + kZ)(i\xi + j\eta + k\zeta),$$

that is,

$$w + ix + jy + kz = \xi X + \eta Y + \zeta Z + i(\eta Z - \zeta Y) + j(\zeta X - \xi Z) + k(\xi Y - \eta X),$$

as is at once verified by substituting for X, Y, Z their values

$$w\xi + y\zeta - z\eta, \quad w\eta + z\xi - x\zeta, \quad w\zeta + x\eta - y\xi.$$

Observe that

$$Xx + Yy + Zz = w(\xi x + \eta y + \zeta z) = 0.$$

38.   And we hence reduce the multiplication of quaternions to that of vectors. For, considering another quaternion Q', take $U = i\xi + j\eta + k\zeta$ as before, with the further condition that it is at right angles to Q', so that $\xi^2 + \eta^2 + \zeta^2 = 1$, $\xi x + \eta y + \zeta z = 0$, $\xi x' + \eta y' + \zeta z' = 0$; say U is the unit-vector at right angles to the vectors of Q, Q': then we have

$$-UQ' = -(i\xi + j\eta + k\zeta)(w' + ix' + jy' + kz')$$
$$= i(-w'\xi + y'\zeta - z'\eta) + j(-w'\eta + z'\xi - x'\zeta) + k(-w'\zeta + x'\eta - y'\xi),$$

which is a vector, say it is

$$V' = iX' + jY' + kZ';$$

and then

$$Q' = -U^2 Q' = UV' = (i\xi + j\eta + k\zeta)(iX' + jY' + kZ'),$$

that is,

$$w' + ix' + jy' + kz' = -\xi X' - \eta Y' - \zeta Z' + i(\eta Z' - \zeta Y') + j(\zeta X' - \xi Z') + k(\xi Y' - \eta X'),$$

as may be verified by substituting for X', Y', Z' their values

$$-w'\xi + y'\zeta - z'\eta, \quad -w'\eta + z'\xi - x'\zeta, \quad -w'\zeta + x'\eta - y'\xi;$$

and it may be observed that

$$X'x' + Y'y' + Z'z', \quad = -w(\xi x' + \eta y' + \zeta z') = 0.$$

We hence find

$$QQ' = QU \cdot -UQ' = VV' = (iX + jY + kZ)(iX' + jY' + kZ'),$$
$$= -XX' - YY' - ZZ' + i(YZ' - Y'Z) + j(ZX' - Z'X) + k(XY' - X'Y).$$

39. This should, of course, be identically equal to

$$ww' - xx' - yy' - zz'$$
$$+ i(wx' + xw' + yz' - y'z)$$
$$+ j(wy' + yw' + zx' - z'x)$$
$$+ k(wz' + zw' + xy' - x'y),$$

on substituting for X, Y, Z, X', Y', Z' their values as above.

Thus, forming the value of $XX' + YY' + ZZ'$, and adding thereto the expression $(\xi x + \eta y + \zeta z)(\xi x' + \eta y' + \zeta z')$, which is identically $= 0$, we obtain

$$XX' + YY' + ZZ' = (-ww' + xx' + yy' + zz')(\xi^2 + \eta^2 + \zeta^2),$$
$$= -ww' + xx' + yy' + zz';$$

and similarly the other three relations may be verified.

40. The steps are, taking U the unit-vector at right angles to Q, Q', then the vectors V, V' are defined by $V = QU$, $V' = -UQ'$; and we then have $Q = -VU$, $Q' = UV'$, and thence $QQ' = VV'$.

We may look at the process as follows: taking first $Q = -VU$, we have V, U vectors in a determinate plane and at a determinate inclination to each other; taking then $Q' = U'V'$, we have U', V' vectors in a determinate plane and at a determinate inclination to each other; we then rotate the vector-pairs (V, U) and (U', V') each in its own plane and in the senses V to U and U' to V' respectively, in such wise as to bring the vectors U, U' into coincidence (along one or other of the opposite vectors which form the intersection of the two planes), and this being so, and U, U' being unit-vectors, we have $U = U'$, and consequently $QQ' = -VU \cdot U'V' = VV'$ as above.

41. There is a kind of quaternion operator $Q^{-1}(\ )Q$, or, what is the same thing, if $\Lambda = i\lambda + j\mu + k\nu$, $(1-\Lambda)(\ )(1+\Lambda)$, which is the symbol of a rotation; viz. the operand, say the vector $ix + jy + kz$, is to be placed within the vacant $(\ )$, say

$$Q^{-1}(\ )Q.(ix + jy + kz) \quad \text{means} \quad Q^{-1}(ix + jy + kz)Q\,;$$

and this being so, the result is a vector $ix_1 + jy_1 + kz_1$, where the x_1, y_1, z_1 are what the x, y, z become by a rotation of the vector $ix + jy + kz$ about an axis and through an angle determined by the quaternion Q; viz. if as above

$$Q = 1 + i\lambda + j\mu + k\nu,$$

then writing

$$\lambda,\ \mu,\ \nu = \tan\tfrac{1}{2}\theta\cos f, \quad \tan\tfrac{1}{2}\theta\cos g, \quad \tan\tfrac{1}{2}\theta\cos h,$$

where $\cos^2 f + \cos^2 g + \cos^2 h = 1$, and therefore $\lambda^2 + \mu^2 + \nu^2 = \tan^2\tfrac{1}{2}\theta$, then f, g, h are the inclinations of the axis, and θ is the angle of rotation. The actual formula is

$$(1 - i\lambda - j\mu - k\nu)(ix + jy + kz)(1 + i\lambda + j\mu + k\nu) = ix_1 + jy_1 + kz_1,$$

where

$$ix_1 + jy_1 + kz_1 = \frac{1}{1 + \lambda^2 + \mu^2 + \nu^2}\left\{ \begin{array}{l} i\left[(1 + \lambda^2 - \mu^2 - \nu^2)x + 2(\lambda\mu + \nu)y + 2(\lambda\nu - \mu)z\right] \\ + j\left[2(\lambda\mu - \nu)x + (1 - \lambda^2 + \mu^2 - \nu^2)y + 2(\mu\nu + \lambda)z\right] \\ + k\left[2(\nu\lambda + \mu)x + 2(\mu\nu - \lambda)y + (1 - \lambda^2 - \mu^2 + \nu^2)z\right] \end{array} \right\},$$

viz. the form on the right-hand side is

$$i\,(\alpha x + \alpha' y + \alpha'' z)$$
$$+ j\,(\beta x + \beta' y + \beta'' z)$$
$$+ k\,(\gamma x + \gamma' y + \gamma'' z),$$

where the coefficients are those of a rectangular transformation.

42. I call to mind that a binary matrix may be regarded as a quaternion; viz. writing

$$\left|\begin{array}{cc} a, & b \\ c, & d \end{array}\right| = (a+d) - \lambda(a-d)i + (b-c)j - \lambda(b+c)k,$$

where i, j, k are the quaternion imaginaries, and λ is written to denote the $i = \sqrt{(-1)}$ of ordinary analysis, then we at once deduce the equation

$$\left|\begin{array}{cc} a, & b \\ c, & d \end{array}\right| . \left|\begin{array}{cc} a', & b' \\ c', & d' \end{array}\right| = (a,\ b) \overset{(a',\ c'),\ (b',\ d')}{\left|\rule{0pt}{20pt}\right.} = \left|\begin{array}{cc} aa' + bc', & ab' + bd' \\ ca' + dc', & cb' + dd' \end{array}\right|$$

for the product of two matrices.

The Ausdehnungslehre: Grassmann. Art. Nos. 43—63.

43. Grassmann. *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik,* Leipzig, 1844; 2nd edition, with the title *Die Ausdehnungslehre von 1844,* Leipzig, 1878.

—— *Die Ausdehnungslehre, vollständig und in strenger Form bearbeitet,* Berlin, 1862, referred to as *Die Ausdehnungslehre von* 1862.

44. Plane Geometry. The representation of a point is that employed by Möbius in the *Barycentrische Calcul* (1827), viz. considering in the plane three fixed points A_1, A_2, A_3, and in regard to the triangle formed by these points, taking x_1, x_2, x_3 as the areal coordinates of a point x (x_1, x_2, x_3 are equal to the areas of the triangles xA_2A_3, xA_3A_1, xA_1A_2, each divided by the area of the triangle $A_1A_2A_3$; whence $x_1 + x_2 + x_3 = 1$), we write

$$x, \ = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3$$

as the representation of the point x. Here ϵ_1, ϵ_2, ϵ_3 are extraordinaries, or they may also be regarded as points, viz. as the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, or say the points A_1, A_2, A_3 respectively. If the sum $x_1 + x_2 + x_3$, instead of being $= 1$ has any other value $= w$, then $x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3$ represents not a point simpliciter, but a point of the weight w; a point simpliciter thus means a point of the weight 1.

45. Considering then

$$\lambda x, \ = \lambda (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3), \quad (x_1 + x_2 + x_3 = 1),$$

a point of the weight λ; and similarly

$$\mu y, \ = \mu (y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3), \quad (y_1 + y_2 + y_3 = 1),$$

a point of the weight μ; we form herewith a sum $\lambda x + \mu y$,

$$= (\lambda x_1 + \mu y_1) \epsilon_1 + (\lambda x_2 + \mu y_2) \epsilon_2 + (\lambda x_3 + \mu y_3) \epsilon_3,$$

which is a point of the weight $\lambda + \mu$, at the C. G. of the two given points, considered as being of the weights λ and μ respectively; if the weights λ and μ are regarded as indeterminate, then the ratio $\lambda : \mu$ is a variable parameter, and the sum will be any point in the line through the given points. It may be remarked that, if $\lambda + \mu$ be $= 1$, then the sum will be a point simpliciter; in particular,

$$\tfrac{1}{2}(x + y), \ = \tfrac{1}{2}(x_1 + y_1)\epsilon_1 + \tfrac{1}{2}(x_2 + y_2)\epsilon_2 + \tfrac{1}{2}(x_3 + y_3)\epsilon_3,$$

will be the point M, midway between the given points x and y.

Observe that $x - y$ is not properly a point; it may be regarded as a point, weight zero, at an infinite distance.

46. The extraordinaries ϵ_1, ϵ_2, ϵ_3 are assumed to be such that

$$\epsilon_1{}^2 = 0, \quad \epsilon_2{}^2 = 0, \quad \epsilon_3{}^2 = 0, \quad \epsilon_2\epsilon_1 = -\epsilon_1\epsilon_2, \quad \epsilon_3\epsilon_2 = -\epsilon_2\epsilon_3, \quad \epsilon_1\epsilon_3 = -\epsilon_3\epsilon_1;$$

the product of the two points x, y is thus

$$x \cdot y = (x_2y_3 - x_3y_2)\epsilon_2\epsilon_3 + (x_3y_1 - x_1y_3)\epsilon_3\epsilon_1 + (x_1y_2 - x_2y_1)\epsilon_1\epsilon_2;$$

depending on the new extraordinaries $\epsilon_2\epsilon_3$, $\epsilon_3\epsilon_1$, $\epsilon_1\epsilon_2$. Grassmann further assumes $\epsilon_1\epsilon_2\epsilon_3 = 1$, and he calls the foregoing combinations the complements (Ergänzungen) of ϵ_1, ϵ_2, ϵ_3, denoting them by $/\epsilon_1$, $/\epsilon_2$, $/\epsilon_3$ respectively; it is better to use new letters, and I call them η_1, η_2, η_3 respectively; we thus have

$$\epsilon_1^2 = 0, \quad \epsilon_2^2 = 0, \quad \epsilon_3^2 = 0,$$

$$\epsilon_2\epsilon_3 = -\epsilon_3\epsilon_2 = \eta_1, \quad \epsilon_3\epsilon_1 = -\epsilon_1\epsilon_3 = \eta_2, \quad \epsilon_1\epsilon_2 = -\epsilon_2\epsilon_1 = \eta_3.$$

But we require further assumptions for the combinations of the η's *inter se* and with the ϵ's; the forms are

$$\eta_1^2 = 0, \quad \eta_2^2 = 0, \quad \eta_3^2 = 0,$$

$$-\eta_2\eta_3 = \eta_3\eta_2 = \epsilon_1; \quad -\eta_3\eta_1 = \eta_1\eta_3 = \epsilon_2, \quad -\eta_1\eta_2 = \eta_2\eta_1 = \epsilon_3;$$

$$\epsilon_1\eta_1 = \eta_1\epsilon_1 = 1, \quad \epsilon_2\eta_2 = \eta_2\epsilon_2 = 1, \quad \epsilon_3\eta_3 = \eta_3\epsilon_3 = 1, \quad \epsilon_1\eta_2 = \eta_2\epsilon_1 = 0, \ \&c.,$$

(viz. each such combination is $= 0$); or, what is the same thing, we have for 1, ϵ_1, ϵ_2, ϵ_3, η_1, η_2, η_3 the multiplication table

	1	ϵ_1	ϵ_2	ϵ_3	η_1	η_2	η_3
1	1	ϵ_1	ϵ_2	ϵ_3	η_1	η_2	η_3
ϵ_1	ϵ_1	0	η_3	$-\eta_2$	1	0	0
ϵ_2	ϵ_2	$-\eta_3$	0	η_1	0	1	0
ϵ_3	ϵ_3	η_2	$-\eta_1$	0	0	0	1
η_1	η_1	1	0	0	0	$-\epsilon_3$	ϵ_2
η_2	η_2	0	1	0	ϵ_3	0	$-\epsilon_1$
η_3	η_3	0	0	1	$-\epsilon_2$	ϵ_1	0

It is proper to remark that in Grassmann's theory there is no interpretation for the general linear symbol

$$\omega + x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3 + y_1\eta_1 + y_2\eta_2 + y_3\eta_3,$$

which presents itself in connexion with the seven extraordinaries $(1, \ \epsilon_1, \ \epsilon_2, \ \epsilon_3, \ \eta_1, \ \eta_2, \ \eta_3)$.

47. It is to be noticed that the symbols are not associative; assuming them to be so, we should have $\epsilon_2\epsilon_3 . \epsilon_3\epsilon_1 = \epsilon_2 . \epsilon_3^2 . \epsilon_1 = 0$, which is inconsistent with $\eta_1\eta_2 = -\epsilon_3$; and so in other cases. But any *three* ϵ's are associative, and hence a product

$$x \cdot y \cdot z, \ = (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3)(y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3)(z_1\epsilon_1 + z_2\epsilon_2 + z_3\epsilon_3),$$

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is associative; in fact, if this be regarded as standing for $(x \cdot y)\, z$, the value is

$$= \{(x_2 y_3 - x_3 y_2)\, \eta_1 + (x_3 y_1 - x_1 y_3)\, \eta_2 + (x_1 y_2 - x_2 y_1)\, \eta_3\}\,(z_1 \epsilon_1 + z_2 \epsilon_2 + z_3 \epsilon_3),$$

which is

$$= (x_2 y_3 - x_3 y_2)\, z_1 + (x_3 y_1 - x_1 y_3)\, z_2 + (x_1 y_2 - x_2 y_1)\, z_3, = \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix};$$

and similarly, regarding it as standing for $x\,(y \cdot z)$, the value is

$$= (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)\, \{(y_2 z_3 - y_3 z_2)\, \eta_1 + (y_3 z_1 - y_1 z_3)\, \eta_2 + (y_1 z_2 - y_2 z_1)\, \eta_3\},$$

which is

$$= x_1\,(y_2 z_3 - y_3 z_2) + x_2\,(y_3 z_1 - y_1 z_3) + x_3\,(y_1 z_2 - y_2 z_1), = \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

as before. The product of the three points is thus a scalar, viz. it is equal to the area of the triangle formed by the three points divided by that of the triangle $A_1 A_2 A_3$.

48. But a product $x \cdot Y \cdot z$,

$$= (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)\,(Y_1 \eta_1 + Y_2 \eta_2 + Y_3 \eta_3)\,(z_1 \epsilon_1 + z_2 \epsilon_2 + z_3 \epsilon_3),$$

is not associative, and has thus no meaning until the grouping of the factors is determined. In fact, $(x \cdot Y)\, z$ will be

$$= (x_1 Y_1 + x_2 Y_2 + x_3 Y_3)\,(z_1 \epsilon_1 + z_2 \epsilon_2 + z_3 \epsilon_3),$$

which denotes the point z_1 with a weight $= x_1 Y_1 + x_2 Y_2 + x_3 Y_3$; whereas $x\,(Y \cdot z)$ will be

$$= (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)\,(y_1 Z_1 + y_2 Z_2 + y_3 Z_3),$$

which denotes the point x_1 with a weight $= y_1 Z_1 + y_2 Z_2 + y_3 Z_3$.

Hence also a product $x \cdot y \cdot z \cdot w$,

$$= (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)\,(y_1 \epsilon_1 + y_2 \epsilon_2 + y_3 \epsilon_3)\,(z_1 \epsilon_1 + z_2 \epsilon_2 + z_3 \epsilon_3)\,(w_1 \epsilon_1 + w_2 \epsilon_2 + w_3 \epsilon_3),$$

of four factors is not associative, and has thus no meaning until the grouping of the factors is determined. Grassmann attributes to such a product the value $= 0$; but this is not a value corresponding to any grouping of the factors, and the equation $x \cdot y \cdot z \cdot w = 0$ can only be regarded as an independent assumption.

For the product of a point into itself, or say the square of a point, we have

$$x^2 = (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)^2, = (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)\,(x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3), = 0,$$

viz. this is $=$ zero.

49. The product of two points x and y is defined to be the "Strecke", the finite line or stroke (xy), considered as drawn from x to y, say we have $x \cdot y = (xy)$; observe

that $y \cdot x = (yx)$: by what precedes $y \cdot x + x \cdot y = 0$, whence also $(yx) + (xy) = 0$, which is an equation in the addition of strokes. Hence, if as before, the points x, y are $x = x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3$, and $y = y_1 \epsilon_1 + y_2 \epsilon_2 + y_3 \epsilon_3$, then for the stroke (xy) we have

$$(xy) = (x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3)(y_1 \epsilon_1 + y_2 \epsilon_2 + y_3 \epsilon_3),$$

$$= (x_2 y_3 - x_3 y_2) \eta_1 + (x_3 y_1 - x_1 y_3) \eta_2 + (x_1 y_2 - x_2 y_1) \eta_3,$$

which is, in fact,

$$= (A_1 xy \cdot \eta_1 + A_2 xy \cdot \eta_2 + A_3 xy \cdot \eta_3) \div A_1 A_2 A_3,$$

if $A_1 xy$, $A_2 xy$, $A_3 xy$, $A_1 A_2 A_3$ denote the areas of the triangles formed by the points (A_1, x, y), (A_2, x, y), (A_3, x, y), (A_1, A_2, A_3) respectively. These triangles remain unaltered if the stroke (xy) is slid along the indefinite line joining the original points x, y, the absolute distance xy being unaltered; that is, strokes of equal length upon the same line are regarded as identical. Hence also strokes on the same line may be added together by adding their lengths.

50. Consider two strokes (xy), (xz), (fig. 4), having a common point x, then completing the parallelogram, we prove that the sum $(xy) + (xz)$ is equal to the diagonal (xw).

Fig. 4.
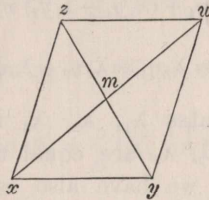
We have, in fact,

$$x + w = y + z,$$

and thence

$$x(x + w) = x(y + z),$$

that is,

$$x^2 + x \cdot w = x \cdot y + x \cdot z,$$

or, since $x^2 = 0$ and $x \cdot y = (xy)$, &c., this is

$$(xw) = (xy) + (xz),$$

the required formula for the addition of the strokes.

51. The same thing appears thus: writing

$$x = x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3,$$

$$y = y_1 \epsilon_1 + y_2 \epsilon_2 + y_3 \epsilon_3,$$

$$z = z_1 \epsilon_1 + z_2 \epsilon_2 + z_3 \epsilon_3,$$

we have

$$(xy) = (x_2 y_3 - x_3 y_2) \eta_1 + (x_3 y_1 - x_1 y_3) \eta_2 + (x_1 y_2 - x_2 y_1) \eta_3,$$

$$(xz) = (x_2 z_3 - x_3 z_2) \eta_1 + (x_3 z_1 - x_1 z_3) \eta_2 + (x_1 z_2 - x_2 z_1) \eta_3.$$

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Or, putting for a moment

$$m_1, \ m_2, \ m_3 = \tfrac{1}{2}\,(y_1 + z_1), \quad \tfrac{1}{2}\,(y_2 + z_2), \quad \tfrac{1}{2}\,(y_3 + z_3),$$

we have for the point m, midway between y and z, the equation $m = m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3$; and then

$$(xy) + (xz) = 2\,\{(x_2 m_3 - x_3 m_2)\,\eta_1 + (x_3 m_1 - x_1 m_3)\,\eta_2 + (x_1 m_2 - x_2 m_1)\,\eta_3\}$$

$$= 2\,(xm), \quad = xw,$$

which is the theorem in question.

52. Recurring to the foregoing equation $x + w = y + z$, we deduce

$$xz - yw = zw - xy = (yzw) = (zwx) = -\,(wxy) = -\,(xyz),$$

if for shortness $(yzw) = yz + zw + wy$, and similarly for (zwx), (wxy), and (xyz). Observe here that $xz - yw$ and $zw - xy$ are each of them equal to the difference between two equal and parallel strokes; the strokes yz, zw, wy are the sides (in order) of a triangle; quâ forces, the expressions in question represent each of them a couple.

53. We have thus arrived at the representation of a stroke

$$(xy) = (x_2 y_3 - x_3 y_2)\,\eta_1 + (x_3 y_1 - x_1 y_3)\,\eta_2 + (x_1 y_2 - x_2 y_1)\,\eta_3,$$

or say

$$\lambda = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3,$$

where the meaning of the coordinates λ_1, λ_2, λ_3 is such that, if x, y are the two extremities of the stroke, then λ_1, λ_2, λ_3 are equal to the areas $A_1 xy$, $A_2 xy$, $A_3 xy$, each divided by the area $A_1 A_2 A_3$. And we have also established the rule for the addition of strokes: it will be noticed that this has been done without the aid of any expression for the length of the stroke.

54. *Metrical relations.* For the expression of the distance of two points or length of a stroke, and of the inclinations of a stroke to the sides of the fundamental triangle, &c., we require the values of the angles and sides of this triangle $A_1 A_2 A_3$, say the angles are A_1, A_2, A_3, and the sides are $= R \sin A_1$, $R \sin A_2$, $R \sin A_3$ respectively; R is thus equal to the diameter of the circumscribed circle. Moreover, considering a stroke xy, if through x we draw lines $x\theta_1$, $x\theta_2$, $x\theta_3$ in the senses $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ respectively, then taking the angles $yx\theta_1$, $yx\theta_2$, $yx\theta_3$ each of them in the same sense, or say each right-handedly, we call these angles θ_1, θ_2, θ_3 respectively, and take also ρ, $= (xy)$ for the length of the stroke. This being so, and if the two extremities are

$$x = x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3, \quad \text{and} \quad y = y_1\epsilon_1 + y_2\epsilon_2 + y_3\epsilon_3,$$

then we have, first,

$$\theta_2 - \theta_3 = \pi - A_1, \quad \theta_3 - \theta_1 = \pi - A_2, \quad \theta_1 - \theta_2 = -\,\pi - A_3,$$

giving

$$A_1 + A_2 + A_3 = \pi,$$

and

$$\sin A_1, \quad \sin A_2, \quad \sin A_3 = \sin\,(\theta_2 - \theta_3), \quad \sin\,(\theta_3 - \theta_1), \quad \sin\,(\theta_1 - \theta_2).$$

And then

$$y_1 = x_1 + \frac{\rho}{R}\frac{\sin\theta_1}{\sin A_2 \sin A_3}, \quad y_2 = x_2 + \frac{\rho}{R}\frac{\sin\theta_2}{\sin A_3 \sin A_1}, \quad y_3 = x_3 + \frac{\rho}{R}\frac{\sin\theta_3}{\sin A_1 \sin A_2}.$$

As regards these last equations, writing for a moment p_1, p_2, p_3 for the perpendicular distances of x, and q_1, q_2, q_3 for those of y from the sides of the triangle, then we have q_1, q_2, $q_3 = p_1 + \rho\sin\theta_1$, $p_2 + \rho\sin\theta_2$, $p_3 + \rho\sin\theta_3$, and we thence easily derive the equations in question.

55. *Expression for length of a stroke.* We hence, by a somewhat complicated analytical process, find an expression for ρ in terms of the coordinates (x_1, x_2, x_3), and (y_1, y_2, y_3), which enter into it through the combinations

$$x_2 y_3 - x_3 y_2, \quad x_3 y_1 - x_1 y_3, \quad x_1 y_2 - x_2 y_1.$$

In fact, writing for shortness

$$\frac{x_1}{\sin A_1}, \quad \frac{x_2}{\sin A_2}, \quad \frac{x_3}{\sin A_3} = K_1, \ K_2, \ K_3,$$

we have

$$(x_2 y_3 - x_3 y_2)\sin A_1 = \frac{\rho}{R}(K_2 \sin\theta_3 - K_3 \sin\theta_2),$$

$$(x_3 y_1 - x_1 y_3)\sin A_2 = \frac{\rho}{R}(K_3 \sin\theta_1 - K_1 \sin\theta_3),$$

$$(x_1 y_2 - x_2 y_1)\sin A_3 = \frac{\rho}{R}(K_1 \sin\theta_2 - K_2 \sin\theta_1);$$

and hence also, if $(* \dsum \xi, \eta, \zeta)^2$ denote the quadric function

$$(1, 1, 1, -\cos A_1, -\cos A_2, -\cos A_3 \dsum \xi, \eta, \zeta)^2,$$

we have

$$(* \dsum (x_2 y_3 - x_3 y_2)\sin A_1, \ldots)^2$$

$$= \frac{\rho^2}{R^2}(* \dsum K_2 \sin\theta_3 - K_3 \sin\theta_2, \ K_3 \sin\theta_1 - K_1 \sin\theta_3, \ K_1 \sin\theta_2 - K_2 \sin\theta_1)^2,$$

where the quadric function on the right-hand side is in fact

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2,$$

that is,

$$= (x_1 + x_2 + x_3)^2, \ = 1;$$

and we thus finally obtain

$$\rho^2 = R^2 (* \dsum (x_2 y_3 - x_3 y_2)\sin A_1, \ (x_3 y_1 - x_1 y_3)\sin A_2, \ (x_1 y_2 - x_2 y_1)\sin A_3)^2,$$

the required formula for the length of the stroke xy.

56. To effect the foregoing reduction of the quadric function

$$(* \dsum K_2 \sin\theta_3 - K_3 \sin\theta_2, \ K_3 \sin\theta_1 - K_1 \sin\theta_3, \ K_1 \sin\theta_2 - K_2 \sin\theta_1)^2,$$

observe that the coefficient herein of K_1^2 is

$$= \sin^2 \theta_2 + \sin^2 \theta_3 + 2 \cos A_1 \sin \theta_2 \sin \theta_3,$$

where $\theta_2 - \theta_3 = \pi - A_1$, and thence

$$\cos \theta_2 \cos \theta_3 + \sin \theta_2 \sin \theta_3 = - \cos A_1 ;$$

that is, we have

$$\cos A_1 + \sin \theta_2 \sin \theta_3 = - \cos \theta_2 \cos \theta_3 ,$$

and thence

$$1 - \sin^2 \theta_2 - \sin^2 \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3 = \cos^2 A_1 + 2 \cos A_1 \sin \theta_2 \sin \theta_3 + \sin^2 \theta_2 \sin^2 \theta_3 ;$$

whence the coefficient in question is $= \sin^2 A_1$. And finding in like manner the coefficients of K_2^2, K_3^2, $K_2 K_3$, &c., the whole function is, as already mentioned,

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2, \text{ that is, } = 1.$$

57. *New representation of a stroke.* Representing the stroke in the form

$$\lambda = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3,$$

we have only to replace $x_2 y_3 - x_3 y_2$, $x_3 y_1 - x_1 y_3$, $x_1 y_2 - x_2 y_1$ by their values $= \lambda_1$, λ_2, λ_3 respectively, viz. for the length ρ of the stroke, we have

$$\rho^2 = R^2 (*\text{\char"0292}\lambda_1 \sin A_1, \ \lambda_2 \sin A_2, \ \lambda_3 \sin A_3)^2.$$

Say this equation is

$$\rho^2 = R^2 . \Omega, \text{ then } \rho = R \sqrt{(\Omega)}, \text{ or } 1 = \frac{\rho}{R \sqrt{(\Omega)}},$$

and the stroke may be represented in the form

$$\lambda = \frac{\rho}{R \sqrt{(\Omega)}} (\lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3) ;$$

or, what is the same thing, if the absolute magnitudes of the coordinates λ_1, λ_2, λ_3 are such that

$$(*\text{\char"0292}\lambda_1 \sin A_1, \ \lambda_2 \sin A_2, \ \lambda_3 \sin A_3)^2 = 1,$$

then the expression is

$$\lambda = \frac{\rho}{R} (\lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3),$$

where ρ is the length of the stroke.

58. It has been seen that the coefficients of η_1, η_2, η_3 have the values $A_1 xy$, $A_2 xy$, $A_3 xy$, each divided by $A_1 A_2 A_3$; or, what is the same thing, if p_1, p_2, p_3 are the lengths of the perpendiculars on the stroke from the points A_1, A_2, A_3 respectively, then these coefficients are $(\frac{1}{2}\rho p_1, \frac{1}{2}\rho p_2, \frac{1}{2}\rho p_3)$, each divided by $A_1 A_2 A_3$; or, what is the same thing, the perpendicular distances p_1, p_2, p_3 are

$$= \frac{2 A_1 A_2 A_3}{R \sqrt{\Omega}} (\lambda_1, \ \lambda_2, \ \lambda_3)$$

respectively, where $A_1 A_2 A_3$ is the area of the fundamental triangle.

59. *Expression for the mutual inclination of two strokes.* In connexion with the stroke $\lambda = \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3$, considering a stroke $\mu = \mu_1 \eta_1 + \mu_2 \eta_2 + \mu_3 \eta_3$, suppose the inclinations hereof to the axes are ϕ_1, ϕ_2, ϕ_3, then

$$\phi_2 - \phi_3 = \pi - A_1, \quad \phi_3 - \phi_1 = \pi - A_2, \quad \phi_1 - \phi_2 = -\pi - A_3,$$

and we have

$$\theta_1 - \phi_1 = \theta_2 - \phi_2 = \theta_3 - \phi_3,$$

viz. each of these equal angles is the inclination of the two strokes to each other, say this angle is $= \delta$. The expression for this angle is

$$\cos \delta = (*\!\smallint\!\lambda_1 \sin A_1, \ \lambda_2 \sin A_2, \ \lambda_3 \sin A_3 \smallint\!\mu_1 \sin A_1, \ \mu_2 \sin A_2, \ \mu_3 \sin A_3)$$

$$\div \sqrt{\{(1, \ldots \smallint\lambda_1 \sin A_1, \ \lambda_2 \sin A_2, \ \lambda_2 \sin A_3)^2\}} \cdot \sqrt{\{(1, \ldots \smallint\mu_1 \sin A_1, \ \mu_2 \sin A_2, \ \mu_3 \sin A_3)^2\}}.$$

In fact, considering (as we may do) the two strokes as proceeding from a common point $x = x_1 \epsilon_1 + x_2 \epsilon_2 + x_3 \epsilon_3$, then the function in question is

$$= (*\!\smallint K_2 \sin \theta_3 - K_3 \sin \theta_2, \ K_3 \sin \theta_1 - K_1 \sin \theta_3, \ K_1 \sin \theta_2 - K_2 \sin \theta_1 \smallint K_2 \sin \phi_3 - K_3 \sin \phi_2, \ \ldots),$$

which is

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2 \cos \delta, \quad = (x_1 + x_2 + x_3)^2 \cos \delta, \quad = \cos \delta.$$

60. In verification, observe that the whole coefficient of K_1^2 is

$$= \sin \theta_2 \sin \phi_2 + \sin \theta_3 \sin \phi_3 + \cos A_1 (\sin \theta_2 \sin \phi_3 + \sin \theta_3 \sin \phi_2),$$

where $\theta_2 - \theta_3 = \pi - A_1$, $\phi_2 - \phi_3 = \pi - A_1$. Hence

$$\sin \theta_2 = - \sin (\theta_3 - A_1), \quad \sin \phi_2 = - \sin (\phi_3 - A_1) ;$$

and the expression in question is

$$= \quad \sin (\theta_3 - A_1) \sin (\phi_3 - A_1)$$

$$- \cos A_1 \sin (\theta_3 - A_1) \sin \phi_3,$$

$$- \cos A_1 \sin (\phi_3 - A_1) \sin \theta_3,$$

$$+ \sin \theta_3 \sin \phi_3,$$

which, expanding the sines, and writing the last term in the form

$$- \sin \theta_3 \sin \phi_3 (\cos^2 A_1 + \sin^2 A_1)$$

is

$$= \quad \cos^2 A_1 (\sin \theta_3 \sin \phi_3 - \sin \theta_3 \sin \phi_3 - \sin \theta_3 \sin \phi_3 + \sin \theta_3 \sin \phi_3)$$

$$+ \sin^2 A_1 (\cos \theta_3 \cos \phi_3 + \sin \theta_3 \sin \phi_3)$$

$$+ \cos A_1 \sin A_1 (- \sin \theta_3 \cos \phi_3 - \sin \phi_3 \cos \theta_3 + \sin \phi_3 \cos \theta_3 + \sin \theta_3 \cos \phi_3),$$

$$= \quad \sin^2 A_1 \cos (\theta_3 - \phi_3), \quad = \sin^2 A_1 \cos \delta,$$

and reducing in like manner the coefficients of K_2^2, K_3^2, $K_2 K_3$, &c., the whole expression becomes

$$= (K_1 \sin A_1 + K_2 \sin A_2 + K_3 \sin A_3)^2 \cos \delta, \quad = \cos \delta \text{ as above.}$$

61. The foregoing formulæ agree with the theorem, No. 50, that the sum of the strokes

$$\lambda, = \frac{\rho}{R\sqrt{(\Omega)}}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3),$$

and

$$\mu, = \frac{\sigma}{R\sqrt{(\Upsilon)}}(\mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3),$$

is a stroke

$$= \frac{\tau}{R\sqrt{(\Pi)}}(\nu_1\eta_1 + \nu_2\eta_2 + \nu_3\eta_3),$$

which is the diagonal of the parallelogram constructed with the given strokes λ and μ; the proof is a little simplified by assuming (as is allowable) $\Omega = 1$ and $\Upsilon = 1$, in which case we have

$$\lambda + \mu = \frac{1}{R}\{(\rho\lambda_1 + \sigma\mu_1)\eta_1 + (\rho\lambda_2 + \sigma\mu_2)\eta_2 + (\rho\lambda_3 + \sigma\mu_3)\eta_3\},$$

from which the length and inclination may be calculated.

As already appearing, the product x, y, z of any three points is the scalar

$$\begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

which is equal to the area of the triangle xyz, divided by that of the triangle $A_1A_2A_3$. We have in like manner the product of a point x into a stroke λ, viz. this is

$$= (x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3)(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3),$$

which is $= x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3$, the two factors being in this case commutative; the value is equal to the area of the triangle formed by the point and the stroke, divided by the area $A_1A_2A_3$. Of course, if in the one case the three points are in a line, or if in the other the point is in the line of the stroke, then the product is $= 0$.

62. We have yet to consider the product of two strokes; say these are

$$\lambda = \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3, \quad \text{and} \quad \mu = \mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3,$$

then we have

$$\lambda.\mu = (\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3)(\mu_1\eta_1 + \mu_2\eta_2 + \mu_3\eta_3)$$
$$= -\{(\lambda_2\mu_3 - \lambda_2\mu_2)\epsilon_1 + (\lambda_3\mu_1 - \lambda_1\mu_3)\epsilon_2 + (\lambda_1\mu_2 - \lambda_2\mu_1)\epsilon_3\},$$

which is a point regarded as having weight. If we take the two strokes to be

$$\lambda = (xy) = (x_2y_3 - x_3y_2)\eta_1 + (x_3y_1 - x_1y_3)\eta_2 + (x_1y_2 - x_2y_1)\eta_3,$$

and

$$\mu = (xz) = (x_2z_3 - x_3z_2)\eta_1 + (x_3z_1 - x_1z_3)\eta_2 + (x_1z_2 - x_2z_1)\eta_3,$$

then the product is

$$\lambda \cdot \mu = -\left(x_1\epsilon_1 + x_2\epsilon_2 + x_3\epsilon_3\right) \begin{vmatrix} x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \\ z_1, & z_2, & z_3 \end{vmatrix},$$

viz. this is the common point x, with a weight $=$ area of triangle $xyz \div$ area $A_1A_2A_3$, or say $=$ the area of triangle formed by the two strokes $\div$ area $A_1A_2A_3$. The product is non-commutative: we, in fact, have $\lambda \cdot \mu = -\mu \cdot \lambda$.

63. The product of three strokes is associative; and it is a scalar. If the three strokes are the sides of a triangle LMN, then the value is $= (LMN)^2 \div (A_1A_2A_3)^2$, where LMN and $A_1A_2A_3$ denote the areas of the triangle LMN and the triangle $A_1A_2A_3$ respectively; and if any stroke instead of being a side of the triangle LMN be a part only of this side, then the value is diminished proportionally; hence the value is $= \dfrac{\rho\sigma\tau}{MN \cdot NL \cdot LM}\{(LMN)^2 \div (A_1A_2A_3)^2\}$, where ρ, σ, τ are the lengths of the three strokes, MN, NL, LM the lengths of the sides of the triangle LMN, and LMN, $A_1A_2A_3$ are the areas as above.

64. It is to be remarked that the form

$$(* \, \llbracket \xi, \ \eta, \ \zeta \, \rrbracket)^2, \ = (1, \ 1, \ 1, \ -\cos A_1, \ -\cos A_2, \ -\cos A_3 \, \llbracket \xi, \ \eta, \ \zeta \, \rrbracket)^2$$

is the product of two linear factors; it is

$$= (\xi - \eta \cos A_3 - \zeta \cos A_2)^2 + (\eta \sin A_3 - \zeta \sin A_2)^2,$$

and thus

$$= (\xi - \eta e^{iA_3} - \zeta e^{-iA_2})(\xi - \eta e^{-iA_3} - \zeta e^{iA_2}).$$

This corresponds to the theorem that the distance of two points is $= 0$, when the line joining them passes through one of the circular points at infinity; the coordinates of one of these points may be written under the three equivalent forms

$$\begin{aligned} x_1 : x_2 : x_3 = \quad \sin A_1 \quad &: -\sin A_2 e^{-iA_3} : -\sin A_3 e^{-iA_2}, \\ = -\sin A_1 e^{-iA_3} : \quad \sin A_2 \quad &: -\sin A_3 e^{iA_1}, \\ = -\sin A_1 e^{\ iA_2} : -\sin A_2 e^{-iA_1} &: \quad \sin A_3, \end{aligned}$$

and those of the other are of course obtained therefrom by the change of i into $-i$.

(*To be continued*)*.

[* This paper remains incomplete; no continuation appears to have been prepared by Professor Cayley.]