863.

NOTE ON THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS.

[From *Crelle's Journal der Mathem.*, t. CI. (1887), pp. 209—213.]

THE theorem V. given by Fuchs in the memoir "Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten," *Crelle's Journal*, t. LXVIII. (1868), pp. 354—385 (see p. 374) for the purpose of deciding whether the integrals belonging to a group of roots of the "determinirenden Fundamentalgleichung" (or as I call it, the Indicial equation) do or do not involve logarithms, may I think be exhibited in a clearer form.

Starting from the differential equation

$$P(y), \quad = p_0 \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \dots + p_m y, \quad = 0,$$

of the order m, then if X be any function of x not satisfying the differential equation, we can at once form a differential equation of the order $m+1$, satisfied by all the solutions of the differential equation, and having also the solution $y = X$; the required equation is in fact

$$\partial_x P(y) \,.\, P(X) - P(y) \,.\, \partial_x P(X) = 0.$$

This I call the augmented equation.

I recall that the equation $P(y) = 0$, considered by Fuchs, is an equation having for each singular point $x = a$, m regular integrals, viz. the coefficients $p_0, p_1, \dots, p_m$ have the forms $q_0(x-a)^m$, $q_1(x-a)^{m-1}, \dots, q_m$, where $q_0, q_1, \dots, q_m$ are rational and integral functions of $x - a$, q_0 not vanishing for $x = a$, and the other functions $q_1, q_2, \dots, q_m$ not in general vanishing for $x = a$. Writing $y = (x-a)^\theta$, we obtain

$$P(x-a)^\theta = I(\theta)(x-a)^\theta + \text{higher powers of } (x-a),$$

where $I(\theta)$, the coefficient of the lowest power of $(x - a)$, is a function of θ of the order m, which I call the indicial coefficient; and equating it to zero, we have $I(\theta) = 0$, the determinirende Fundamentalgleichung, or Indicial equation, being an equation of the order m. If the roots of this equation are such that no two of them are equal or differ only by an integer number, then we have m particular integrals each of them of the form

$$y=(x-a)^r + \text{higher powers of } (x-a),$$

where r is any root of the indicial equation: but if we have in the indicial equation a group of λ roots $r_1, r_2, \ldots, r_\lambda$, such that the difference of each two of them is either zero or an integer, then the integrals which correspond to these roots involve or may involve logarithms; in particular, if any two of the roots are equal, the integrals for the group will involve logarithms.

Consider now the differential equation $P(y)=0$ in reference to the singular point $x=a$ as above, and writing $X=(x-a)^\epsilon f$ where ϵ is in the first instance arbitrary, and f is a rational and integral function of $x-a$ not vanishing for $x=a$, we form the augmented equation which, observing that we have in general $P(X)=(x-a)^\epsilon Q$, Q a rational and integral function of $x-a$ not vanishing for $x=a$, and dividing the whole equation by $(x-a)^{\epsilon-1}$, may be written

$$\partial_x P(y) \,.\, (x-a)\, Q - P(y)\, \{\epsilon Q + (x-a)\,\partial_x Q\} = 0,$$

an equation of the same form as the original equation (but of the order $m+1$ instead of m), and having an indicial equation

$$(\theta-\epsilon)\, I(\theta) = 0.$$

In fact, writing as before $y=(x-a)^\theta$, we have in $\partial_x P(y) \,.\, (x-a)\, Q$ the term of lowest order $\theta I(\theta)\, Q_0 (x-a)^\theta$ and in $P(y) \,.\, \epsilon Q$ the term of lowest order $\epsilon I(\theta)\, Q_0 (x-a)^\theta$, whereas in $P(y)(x-a)\,\partial_x Q$ the term of lowest order is $(x-a)^{\theta+1}$; the indicial equation is thus as just found.

If however ϵ be equal to a root of the indicial equation $I(\theta)=0$, then instead of $P(X)=(x-a)^\epsilon Q$, we have $P(X)=(x-a)^\mu Q$, where the index μ is $=\epsilon+$ a positive integer, and where the value of the difference $\mu-\epsilon$ may depend upon the determination of the function f in the expression $(x-a)^\epsilon f$. The indicial equation for the augmented equation is in this case $(\theta-\mu)\, I(\theta)=0$.

If the indicial equation $I(\theta)=0$ of the given differential equation has a group of roots $r_1, r_2, \ldots, r_\lambda$, the difference of any two of these roots being zero or an integer, then taking $\epsilon=$ any one of these roots, the augmented equation will have a group of roots $(\mu, r_1, r_2, \ldots, r_\lambda)$.

If any two of the roots $r_1, r_2, \ldots, r_\lambda$ are equal, the group of integrals $u_1, u_2, \ldots, u_\lambda$ will involve logarithms: the question only arises when these roots are unequal, and taking them to be so, the theorem V. is in effect as follows: "If by taking $\epsilon=$ some one of the roots $r_1, r_2, \ldots, r_\lambda$, and by a proper determination of the function f we can make μ to be $=$ one of the same roots $r_1, r_2, \ldots, r_\lambda$, then the group of integrals $u_1, u_2, \ldots, u_\lambda$ will involve logarithms; but if μ cannot be made $=$ one of the roots $r_1, r_2, \ldots, r_\lambda$, then the group of integrals will be free from logarithms."

As an example, I consider the equation

$$P(y) = (x^2 - x^4)\frac{d^2y}{dx^2} - 2x^3\frac{dy}{dx} - (n^2+n)\, y = 0.$$

This is Legendre's equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+(n^2+n)y=0$, with $\frac{1}{x}$ substituted for x, so that, instead of a singular point $x=\infty$, there may be a singular point $x=0$. Attending to the singular point $x=0$, we have $P(x^\theta)=(\theta^2-\theta-n^2-n)x^\theta+$ higher powers, so that the indicial equation $I(\theta)=0$ is $\theta^2-\theta-n^2-n=0$, that is, $(\theta+n)(\theta-n-1)=0$, or we have the roots $-n$, $n+1$, which differ by an integer, and thus form a group, if n be = an integer, or be = an integer $-\frac{1}{2}$; to fix the ideas, say that the roots are $-p$, $p+1$ or else $-p+\frac{1}{2}$, $p+\frac{1}{2}$ where p is a positive integer.

Writing for greater convenience $x^\epsilon f=x^\epsilon+F$, where F is a sum of powers of x higher than ϵ, we find without difficulty

$$P(x^\epsilon f)=x^\epsilon\{(\epsilon+n)(\epsilon-n-1)-(\epsilon^2+\epsilon)x^2+(x^2-x^4)x^{-\epsilon}F''-(n^2+n)x^{-\epsilon}F\}$$

which, so long as ϵ remains arbitrary, is of the form $x^\epsilon Q$, $Q=(\epsilon+n)(\epsilon-n-1)+$ powers of x; if however ϵ be a root of the indicial equation, for instance, if $\epsilon=-n$, then the expression in brackets { } contains at any rate the factor x, so that the form is $P(x^{-n}f)=x^\mu Q$, where μ is $=-n+1$ at least; we can however, by a proper determination of the function f, make μ acquire a larger value.

For instance, suppose $-n$, $n+1=-2$, 3; $\epsilon=-n=-2$, and assume

$$x^\epsilon f=x^{-2}+Bx^{-1}+Cx^0+Dx^1+Ex^2+Fx^3+Gx^4+\dots$$

To calculate $P(x^\epsilon f)$, we have

	x^{-2}	x^{-1}	x^0	x^1	x^2	x^3	x^4	$x^5\dots$
	6	$2B$	$0C$	$0D$	$2E$	$6F$	$12G$	$20H\dots$
			-6	$-2B$	$-0C$	$-0D$	$-2E$	$-6F$
			$+4$	$+2B$	$-0C$	$-2D$	$-4E$	$-6F$
	-6	$-6B$	$-6C$	$-6D$	$-6E$	$-6F$	$-6G$	$-6H.$
$P(x^\epsilon f)=$	0	$-4B$	$-6C$	$-6D$	$-4E$	$0F$	$6G$	$14H\dots$
			-2			$-2D$	$-6E$	$-12F\dots$

Hence if B not $=0$, we have $\mu=-1$; if $B=0$, $-6C-2$ not $=0$, we have $\mu=0$; if $B=0$, $-6C-2=0$, D not $=0$, we have $\mu=1$; if $B=0$, $-6C-2=0$, $D=0$, but E not $=0$, we have $\mu=2$; if $B=0$, $-6C-2=0$, $D=0$, $E=0$, then the coefficient of x^3, $=0F-2D$, is $=0$, and we have not $\mu=3$, but $\mu=4$ at least, viz. μ will be $=4$, if $6G-6E=0$, that is, if $G=0$; but leaving F arbitrary, we can by giving proper values to the subsequent coefficients H, I, &c., make μ to be $=5$ or any larger integer value. The values of μ are thus $=-1$, 0, 1, 2, 4, 5, ..., and we see that the group $(\mu, -n, n+1)$, that is, $(\mu, -2, 3)$, does not in any case contain two equal indices. Starting from the value $\epsilon=3$, the value of μ is >3, and thus here also the group $(\mu, -2, 3)$ does not contain two equal indices.

The conclusion from the theorem thus is that the integrals u_1, u_2, belonging to the roots -2, 3, do not involve logarithms: and in precisely the same manner, it appears that the integrals, belonging to the two roots $-p$, $p+1$ (p any positive integer), do not involve logarithms: this is right, for the integrals are, in fact, the Legendrian functions of the first and second kinds P_p and Q_p, with only $\frac{1}{x}$ written therein instead of x.

Similarly, if for instance $-n,\ n+1=-\frac{1}{2},\ \frac{3}{2}$, then, if $\epsilon=-n=-\frac{1}{2}$, assuming

$$x^\epsilon f = x^{-\frac{1}{2}} + Bx^{\frac{1}{2}} + Cx^{\frac{3}{2}} + Dx^{\frac{5}{2}} + Ex^{\frac{7}{2}} + \ldots,$$

we have

	$x^{-\frac{1}{2}}$	$x^{\frac{1}{2}}$	$x^{\frac{3}{2}}$	$x^{\frac{5}{2}}$	$x^{\frac{7}{2}}\ldots$
	$\frac{3}{4}$	$-\frac{1}{4}B$	$\frac{3}{4}C$	$\frac{15}{4}D$	$\frac{35}{4}E$
			$-\frac{3}{4}$	$+\frac{1}{4}B$	$-\frac{3}{4}C$
			$+1$	$-B$	$-3C$
	$-\frac{3}{4}$	$-\frac{3}{4}B$	$-\frac{3}{4}C$	$-\frac{3}{4}D$	$-\frac{3}{4}E$
$P(x^\epsilon f)=$	0	$-B$	$0C$	$3D$	$8E\ldots$
			$+\frac{1}{4}$	$-\frac{3}{4}B$	$-\frac{15}{4}C\ldots$

We have here if B not $=0$, $\mu=\frac{1}{2}$; but if $B=0$, then we cannot in any way make the coefficient of $x^{\frac{3}{2}}$ to vanish, and consequently $\mu=\frac{3}{2}$. With this last value of μ, the group $(\mu,\ -n,\ n+1)$, that is, $(\mu,\ -\frac{1}{2},\ \frac{3}{2})$, becomes $(\frac{3}{2},\ -\frac{1}{2},\ \frac{3}{2})$ which contains two equal roots, and the conclusion from the theorem thus is that the integrals u_1, u_2, corresponding to the roots $-\frac{1}{2}$, $\frac{3}{2}$, involve logarithmic values. And similarly in general the integrals u_1, u_2, corresponding to the roots $-p+\frac{1}{2}$, $p+\frac{1}{2}$ (p any positive integer), involve logarithmic values: this also is right.

The examples exhibit the true character of the theorem, and show I think that it is a less remarkable one than would at first sight appear: in fact, in working them out, we really ascertain by an actual substitution whether the differential equation can be satisfied by series of powers only, without logarithms. Thus for $n=2$ as above, it appears that the equation is satisfied by the series

$$y = x^{-2} + Bx^{-1} + Cx^0 + Dx^1 + Ex^2 + Fx^3 + Gx^4 + Hx^5 + \ldots,$$

where

$$B=0,\ C=-\tfrac{1}{3},\ D=0,\ E=0,\ F=F,\ G=0,\ H=-\tfrac{6}{7}F, \ldots,$$

that is, by

$$y = x^{-2} + \tfrac{1}{3} + F(x^3 - \tfrac{6}{7}x^5 + \ldots),$$

in other words, that we have the two particular integrals $y=x^{-2}+\frac{1}{3}$, and $y=x^3-\frac{6}{7}x+\ldots$, belonging to the two roots -2, 3 respectively.

Similarly, when $n=\frac{1}{2}$, we cannot satisfy the equation by a series

$$y = x^{-\frac{1}{2}} + Bx^{\frac{1}{2}} + Cx^{\frac{3}{2}} + Dx^{\frac{5}{2}} + \ldots;$$

for in order to satisfy the equation, we must have $B=0$, $C=\infty$; there is thus no series of powers $y=x^{-\frac{1}{2}}+Cx^{\frac{3}{2}}+\ldots$, corresponding to the root $-\frac{1}{2}$: but there is a series $y=x^{\frac{3}{2}}+kx^{\frac{7}{2}}+\ldots$ corresponding to the root $\frac{3}{2}$; and thus the integrals u_1, u_2, corresponding to these roots $-\frac{1}{2}$, $\frac{3}{2}$, involve logarithms.

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