860.

ON BRIOT AND BOUQUET'S THEORY OF THE DIFFERENTIAL EQUATION $F\left(u, \frac{du}{dz}\right)=0$.

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I CONSIDER the theory of a differential equation of the first order as developed in Briot and Bouquet's *Théorie des fonctions doublement périodiques et en particulier des fonctions elliptiques* (8°, 1st ed., Paris, 1859), but I make some substantial variations in the mode of treatment.

I remark that, writing $u=x$, $\frac{du}{dz}=y$, I make the theory to depend altogether upon that of the curve $F(x, y)=0$; viz. the form of my result is, in order that a differential equation of the first order $F\left(u, \frac{du}{dz}\right)=0$ may have a monotropic integral, the curve $F(x, y)=0$ must satisfy certain conditions. Briot and Bouquet give their result (Theorem IV., p. 296) under a somewhat different form, as follows; viz. changing only the word "monodrome" into "monotrope," their theorem is:—

"Pour qu'une équation différentielle du premier ordre de la forme

$$\left(\frac{du}{dz}\right)^m + f_1(u)\left(\frac{du}{dz}\right)^{m-1} + \ldots\ldots + f_m(u) = 0$$

admette une intégrale monotrope: 1° les coefficients $f_1(u)$, $f_2(u)$, ..., $f_m(u)$ doivent être des polynômes entiers en u et, au plus, le premier du second degré, le second du quatrième degré, ..., le dernier du degré $2m$; 2° quand pour une certaine valeur de u l'équation a une racine multiple différente de zéro, $\frac{du}{dz}$ doit rester monotrope par rapport à u; 3° quand pour une certaine valeur u_1 de u, l'équation a une racine multiple égale à zéro, le premier terme du développement de $\frac{du}{dz}$ suivant les puissances croissantes de $(u-u_1)^{1/n}$ doit avoir l'exposant $1-\frac{1}{n}$, si cet exposant est plus petit que l'unité; 4° enfin l'équation différentielle, que l'on déduit de la première en posant $u=\frac{1}{v}$, doit offrir pour $v=0$ les mêmes caractères.—Ces conditions sont suffisantes."

I notice that this may be regarded as a statement referring to the curve

$$F(x, y), = y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

but that in 2° a notion of monotropy is assumed, for y must be a monotropic function of x; and that 4° in effect introduces the new curve $F\left(\frac{1}{x}, -\frac{y}{x^2}\right) = 0$.

1. We have between the variables u and z a differential equation of the first order $F\left(u, \frac{du}{dz}\right) = 0$, F a rational and integral function of u and $\frac{du}{dz}$. The equation determines u as a function of z, and we wish to know whether u is a monotropic function of z. It will not be so if we have a tropical point, viz. a point $z=c$, such that, in the neighbourhood thereof, the value of u is given by a series $u-a=B(z-c)^\beta+\ldots$, in ascending powers of $z-c$, involving fractional powers of $z-c$. Conversely, if there is no tropical point, then u will be a monotropic function of z; but we must consider and exclude not only tropical points at a finite distance, but also the point at infinity, viz. there must not be any development $u=Bz^\beta+\ldots$, in descending powers, involving fractional powers of z.

2. A one-valued function (*eindeutige Function*, or *fonction uniforme* or *bien déterminée*) is monotropic; since there is only one value, the function must, after a circuit described by the point z, recover its original value. But, conversely, a monotropic function is one-valued, that is, no two or more valued function can be monotropic. For, suppose a two-valued function of z, having the values Z_1 and Z_2, is monotropic; there is by hypothesis no tropical point, that is, no point $z=c$, such that the function Z_1, after a circuit described by the point z, instead of recovering its original value, assumes the value Z_2; and the like as to Z_2. But, this being so, it is meaningless to regard Z_1 and Z_2 as two values of the same function; they must be considered as two distinct and separate functions, each of them a one-valued monotropic function. And it thus appears that the two notions, monotropic and one-valued, are in fact equivalent.

3. In what precedes, u and z are regarded as complex variables geometrically represented by means of two real points in their own planes respectively. But we now write $\frac{du}{dz}=y$, $u=x$, whence $F(x, y)=0$, regarding x, y as the coordinates, real or imaginary, of a point of the curve represented by this equation. And this curve has, moreover, to be considered from a non-projective point of view; we have to distinguish between, and separately consider, finite points (x and y each of them

finite), and the points at infinity (x and y either or each of them infinite). And it is moreover necessary to distinguish between, and separately consider, points on the axis of x (or say axial points), and points off the axis of x (say non-axial points). Taking, for convenience, a to denote a finite value which may be $=0$, b a finite value which is not $=0$ (or say, a not $=\infty$, b not $=\infty$ or 0), we have thus the six cases, 1° (a, b); 2° $(a, 0)$; 3° (a, ∞); 4° (∞, b); 5° $(\infty, 0)$; 6° (∞, ∞). I regard also the axes of x and y as horizontal and vertical respectively, and speak of the tangent or element of the curve as horizontal, vertical, or inclined, accordingly.

4. In the neighbourhood of any given point of the curve, we have $y=$ a series of powers of x, the form of the series being different for different classes of points. I write $P(x)$ to denote a series (finite or infinite) $A+Bx+Cx^2+\ldots$, in ascending powers of x; it is to be throughout understood that the leading coefficient A is not $=0$. Of course, $P(x-a)$, $P(x-a)^{1/n}$, $P(x^{-1})$, &c., will denote the like series $A+B(x-a)+\ldots$, $A+B(x-a)^{1/n}+\ldots$, $A+Bx^{-1}+\ldots$, &c. The coefficients after the leading coefficient A may any or all of them vanish; thus $P(x-a)^{1/n}$ extends to denote the series $P(x-a)$ of integer powers, but naturally the former notation is not used unless the series contains fractional powers. We can, by means of the symbol P, express for any given point of the curve the form of the expansion in the neighbourhood thereof; and, attaching to the point the expansion which belongs to it, we may, for instance, speak of a point $y=P(x-a)$; viz. if b is the constant term of the series, then this is a non-axial point (a, b), which is an ordinary (non-singular) point of the curve, and for which the element is not vertical; a like point with the element vertical would be a point $y=b+(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$. Observe that, in the case of a multiple point, there is a separate expansion for each branch through the point, and in thus attaching an expansion to the point we regard the point as belonging to one of these branches; in dealing with a multiple point, it is necessary to consider separately all the different expansions, that is, all the branches through the point. It is clear that for a point (a, b), $(a, 0)$, or (a, ∞) the expression for y depends on $P(x-a)$, or $P(x-a)^{1/n}$; while, for a point (∞, b), $(\infty, 0)$, or (∞, ∞), it depends on $P(x^{-1})$ or $P(x^{-1/n})$; viz. in the two cases respectively, we have a series in ascending powers of $x-a$, or in descending powers of x.

5. In regard to any given point of the curve, substituting for y, x their values $\frac{du}{dz}$, u, and thence forming the reciprocal of $\frac{du}{dz}$, we obtain $\frac{dz}{du}=$ a series in u; viz. this is a series in ascending powers of $(u-a)$, or in descending powers of u. Integrating, we have $z-c=$ a series in u, which series may or may not contain a logarithmic term $\log u$ or $\log(u-a)$; and, reverting the series, we obtain $u=$ a series in $(z-c)$. This result, applicable to the neighbourhood of the point $u=a$, may contain only integer powers of $z-c$, and be accordingly of the form $u=$ a one-valued series in z; and we then say that the point on the curve is a "permissive" point. Or it may contain fractional powers of $z-c$, and be accordingly of the form $u=$ a more-than-one-valued series in z; and we then say that the point on the curve is a "prohibitive" point. The necessary and sufficient condition in order that u may be

a monotropic function of z is:—the points of the curve must be all of them permissive points; or, what is the same thing, there must not be any prohibitive point.

6. I form a complete table of permissive points, as follows:—

n a positive integer $=2$ at least.
$\epsilon = 0, 1,$ or -1,

(a, b), $y = P(x-a)$,

$(a, 0)$, $y = (x-a)P(x-a)$ or $(x-a)^2 P(x-a)$, $\quad y = (x-a)^{1+\epsilon/n} P(x-a)^{1/n}$,

(a, ∞),

(∞, b), $y = P(x^{-1})$,

$(\infty, 0)$,

(∞, ∞), $y = xP(x^{-1})$ or $x^2 P(x^{-1})$, $\quad y = x^{1+\epsilon/n} P\left(\frac{1}{x^{1/n}}\right)$,

viz. it is only a point (a, b), $(a, 0)$, (∞, b) or (∞, ∞) which may be permissive; a point (a, ∞) or $(\infty, 0)$ is prohibitive.

7. A point $y = P(x-a)$ is permissive. We have

$$\frac{du}{dz} = P(u-a),$$

$$\frac{dz}{du} = P(u-a),$$

$$z - c = (u-a)P(u-a),$$

$$u - a = (z-c)P(z-c),$$

the required result. The several steps will be readily understood; the reciprocal of a series $P(u-a)$ is a series of the like form $P(u-a)$; integrating this in regard to u as a series in $u-a$ (no constant being added on the right-hand side), we obtain a series $(u-a)P(u-a)$; and finally, $z-c$ being equal to this expression, we obtain, by reversion, a like expression $u - a = (z-c)P(z-c)$.

We might, for convenience, have written x, u, z in place of $x-a$, $u-a$, $z-c$, respectively. The proof would have run

$$y = P(x),\ \frac{du}{dz} = P(u),\ \frac{dz}{du} = P(u),\ z = uP(u),\ u = zP(z),$$

meaning $u - a = (z-c)P(z-c)$, as above; and, in what follows, this is done accordingly.

It is worth while to make an instance in which the integration can be performed—say we consider the point $(0, 1)$ of the curve $y = \frac{1+x+x^2}{1+2x}$, which point is $y = \frac{1+x+x^2}{1+2x} = P(x)$. We have

$$\frac{du}{dz} = \frac{1+u+u^2}{1+2u},\ \frac{dz}{du} = \frac{1+2u}{1+u+u^2},$$

$$z = \log(1+u+u^2),\ \text{or}\ u + u^2 = (e^z - 1),$$

giving a series $u = zP(z)$; or the point is a permissive point, as in the general case. We have here $u + u^2 = (e^z - 1)$, viz. u is not a monotropic function of z; but this is by reason of a prohibitive point $(-\frac{1}{2}, \infty)$.

8. A point $y = (x-a)P(x-a)$ is permissive.

Writing as above u, z in place of $u-a$, $z-c$, we have

$$\frac{du}{dz} = uP(u), \quad \frac{dz}{du} = \frac{1}{u}P(u), \text{ or say } m\frac{dz}{du} = \frac{1}{u} + Pu,$$

whence

$$mz = \log u + uPu,$$

that is,

$$ue^{uPu} = e^{mz}, \text{ or say } uP(u) = e^{mz},$$

whence

$$u = e^{mz}P(e^{mz}), \text{ a one-valued series.}$$

I take a particular instance, the point (0, 0) for the curve $y = \frac{x - \frac{1}{3}x^3}{1-x^2}$; here

$$\frac{du}{dz} = \frac{u - \frac{1}{3}u^3}{1-u^2}, \text{ or } \frac{dz}{du} = \frac{1-u^2}{u - \frac{1}{3}u^3},$$

$$z = \log(u - \tfrac{1}{3}u^3), \text{ or } u - \tfrac{1}{3}u^3 = e^z;$$

the point is permissive. The equation just obtained shows that u is not a monotropic function of z; but this is by reason of the prohibitive points $(\pm 1, \infty)$.

9. A point $y = (x-a)^2 P(x-a)$ is permissive. Here

$$\frac{du}{dz} = u^2 P(u), \quad \frac{dz}{du} = \frac{1}{u^2}P(u), \quad = \frac{A}{u^2} + \frac{B}{u} + P(u),$$

whence

$$z = -\frac{A}{u} + B\log u + uP(u),$$

where the logarithmic term occurs or does not occur, according as B is not or is $=0$. In the former case, we have

$$e^{z/B} = ue^{-\frac{A}{B}u + uP(u)};$$

in the latter case,

$$z = -\frac{A}{u} + uP(u);$$

and in each case we can, by reversion, express u as a one-valued series in z. We may take as examples the two curves $y = \frac{x^2}{1-x}$ and $y = \frac{x^2}{1-x^2}$; in each case the point (0, 0) is a permissive point, but, by reason of the prohibitive points $(1, \infty)$ and $(\pm 1, \infty)$, u is not a monotropic function of z.

10. As to the other forms of permissive points, the proofs depend upon those for the forms already considered. Thus for

$$y = (x-a)^{1+\epsilon/n} P(x-a)^{1/n}, \text{ or } \frac{du}{dz} = u^{1+\epsilon/n} P(u^{1/n}),$$

writing $v = u^{1/n}$, we find

$$\frac{dv}{dz} = u^{(1+\epsilon)/n} P(u^{1/n}), \; = v^{1+\epsilon} P(v),$$

that is,

$$\frac{dv}{dz} = P(v), \; vP(v), \text{ or } v^2 P(v);$$

in each case $v =$ one-valued series in z, and thence $u = v^n =$ one-valued series in z.

Similarly for

$$y = P(x^{-1}), \; y = xP(x^{-1}), \; y = x^2 P(x^{-1}),$$

or say

$$y = x^{1+\epsilon} P(x^{-1});$$

here

$$\frac{du}{dz} = u^{1+\epsilon} P(u^{-1}),$$

or, writing $v = u^{-1}$, we have

$$\frac{dv}{dz} = u^{\epsilon-1} P(u^{-1}), \; = v^{1-\epsilon} P(v);$$

viz. this is

$$\frac{dv}{dz} = P(v), \; vP(v), \text{ and } v^2 P(v);$$

in each case $v =$ one-valued series in z, and thence $u = v^{-1} =$ one-valued series in z.

And finally, for

$$y = x^{1+\epsilon/n} P\left(\frac{1}{x^{1/n}}\right);$$

here

$$\frac{du}{dz} = u^{1+\epsilon/n} P\left(\frac{1}{u^{1/n}}\right),$$

or, writing $v = u^{-\epsilon/n}$, we have

$$\frac{dv}{dz} = u^{(\epsilon-1)/n} P\left(\frac{1}{u^{1/n}}\right), \; = v^{1-\epsilon} P(v);$$

that is,

$$\frac{dv}{dz} = P(v), \; vP(v), \text{ or } v^2 P(v),$$

as before; $v =$ one-valued series in z, and thence $u = v^{-n} =$ one-valued series in z.

11. There is no difficulty in showing that every point of the curve, not being a permissive point as above, is a prohibitive point. Thus for a point (a, b), if the series for $y - b$ contain any fractional power, say, if we have $y = b + (x-a)^\alpha$, then

$$\frac{du}{dz} = b + u^\alpha, \; \frac{dz}{du} = \frac{1}{b} - \frac{1}{b^2} u^\alpha + \ldots, \; z = \frac{u}{b} - \frac{1}{(\alpha+1) b^2} u^{\alpha+1} + \ldots,$$

whence, reverting, $u = bz +$ series containing the fractional power $z^{\alpha+1}$.

Again, a point $y=(x-a)^m P(x-a)$, $m=3$ or any higher integer value, is prohibitive. Attending only to the leading term, we have $\frac{du}{dz}=u^m$, whence $\frac{dz}{du}=u^{-m}$ or $z=u^{1-m}$, whence, reverting, $u=z^{1/(1-m)}$, which is a fractional power; and this is also the case if m be a negative integer. And similarly, in other cases, the series for u will always contain fractional powers of z.

12. In order that u may be a monotropic function of z, the points of the curve must be all of them permissive, or, what is the same thing, there must be no prohibitive point; we have to consider the geometrical signification of this condition. If we attend first to the ordinary (or non-singular) points of the curve, a non-axial point (a, b) must be a point $y=P(x-a)$, viz. there must be no such point with a vertical element.

An axial point $(a, 0)$ may be $y=(x-a)P(x-a)$, viz. the element may be inclined; or the point may be $y=(x-a)^2 P(x-a)$, viz. the element may be horizontal (but in this case there must be an ordinary or two-pointic contact only); or the point may be $y=(x-a)^{\frac{1}{2}} P(x-a)^{\frac{1}{2}}$, viz. the element may be vertical.

The point must not be (a, ∞), viz. there must be no asymptote parallel to (or coincident with) the axis of y.

The point may be (∞, b), $y=P(x^{-1})$, that is, there may be an ordinary, or osculating, asymptote parallel to the axis of x. But there is no point $(\infty, 0)$, that is, there must be no asymptote coincident with the axis of x.

There may be points (∞, ∞), $y=xP(x^{-1})$, that is, ordinary or osculating asymptotes inclined to the axes; and there may also be points (∞, ∞), $y=x^2P(x^{-1})$, that is, asymptotic parabolas of vertical axis.

13. We have, moreover, conditions as to the singular points; thus, every non-axial multiple point (a, b) must be a point with each branch of the form $y=P(x-a)$, that is, each branch must be an ordinary (non-cuspidal) branch, and must be non-vertical. The axial multiple or singular points $(a, 0)$ may have ordinary (non-cuspidal) branches $y=(x-a)P(x-a)$ and $y=(x-a)^2 P(x-a)$, inclined or horizontal, or else $y=(x-a)^{\frac{1}{2}}P(x-a)^{\frac{1}{2}}$, vertical; and there may also be cuspidal branches $y=(x-a)P(x-a)^{1/n}$, $y=(x-a)^{1+1/n}P(x-a)^{1/n}$, and $y=(x-a)^{1-1/n}P(x-a)^{1/n}$, inclined, horizontal, or vertical.

There must be no singular points (a, ∞); any such point is, in fact, excluded by the condition, no asymptote parallel to or coincident with the axis of y.

There may be multiple points (∞, b), but not $(\infty, 0)$; viz. as above, there may be asymptotes parallel to, but not coincident with, the axis of x; but in this case, the several branches must be each of them an ordinary branch $y=P(x^{-1})$.

Finally, there may be multiple or singular points (∞, ∞), but each branch must be either an ordinary branch $y=xP(x^{-1})$ or $y=x^2P(x^{-1})$, or a cuspidal branch

$$y=xP\left(\frac{1}{x^{1/n}}\right),\quad y=x^{1+1/n}P\left(\frac{1}{x^{1/n}}\right),\quad \text{or}\quad y=x^{1-1/n}P\left(\frac{1}{x^{1/n}}\right).$$

14. The enumeration seems long and difficult. But for a given curve we have, in fact, only to see that there are no ordinary non-axial points (a, b) of vertical element, and to further examine the finite singular points (a, b) and $(a, 0)$, and also the several infinite branches as giving ordinary or else singular points (a, ∞), (∞, b), $(\infty, 0)$, or (∞, ∞), viz. it has to be seen for each infinite branch that the expansion is of the proper form.

15. Writing the equation of the curve in the form

$$f_0(x)\, y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

where $f_0, f_1, \ldots, f_m$ denote rational and integral functions of x, then, if a be a root of the equation $f_0(x) = 0$, there will be on the curve a point (a, ∞), which is prohibitive; $f_0(x)$ is therefore a constant, or, taking it to be $= 1$, the form must be

$$y^m + f_1(x)\, y^{m-1} + \ldots + f_m(x) = 0,$$

where $f_1, f_2, \ldots, f_m$ denote rational and integral functions of x.

It is to be shown that the degrees of these functions are equal respectively to $2, 4, \ldots, m$ at most. Supposing that the degrees are $2, 4, \ldots, m$; then for the points at infinity we find an expansion $y = Ax^\omega + \ldots$, in descending powers of x, by substitution in an equation

$$y^m + \alpha_1 x^2 y^{m-1} + \alpha_2 x^4 y^{m-2} + \ldots + \alpha_n x^{2m} = 0;$$

whence $\omega = 2$, and A is determined by an equation of the order m; there are thus m branches $y = x^2 P(x^{-1})$, corresponding to permissive points. If, however, any one of the functions f be of a higher degree, then it may be shown (but the formal proof is not easily given) that there will be at any rate a branch $y = Ax^\omega + \&c.$, where ω has a value > 2, and we have thus a prohibitive point; hence the degrees must be as stated.

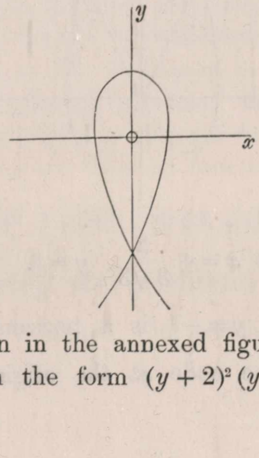
16. As an example of a differential equation with a monotropic solution, take

$$\left(\frac{du}{dz}\right)^3 + 3\left(\frac{du}{dz}\right)^2 + 27u^2 - 4 = 0;$$

we have here the curve

$$y^3 + 3y^2 + 27x^2 - 4 = 0,$$

Fig. 1.



which is a nodal cubic, as shown in the annexed figure. To put the node in evidence, the equation may be written in the form $(y+2)^2(y-1) + 27x^2 = 0$; viz. the node is the point (0, − 2), and the equation of the tangents is $(y+2)^2 - 27x^2 = 0$. The curve besides meets the line $x = 0$ in the point $y = 1$, where the tangent is horizontal, and it meets the axis $y = 0$ in the points $x = \pm \frac{2}{3\sqrt{3}}$, at each of which the tangent is vertical; these two points, as lying on the axis $y = 0$, are thus each of them permissive; and they are the *only* points where the tangent is vertical (in fact, the tangents from the point at infinity on the line $x = 0$ are the two lines $x = \pm \frac{2}{3\sqrt{3}}$, and the line infinity counting twice, as a tangent at an inflexion).

For the points at infinity, we have $y = -3x^{\frac{2}{3}} - 1 - \frac{1}{3}x^{-\frac{2}{3}} + \&c.$, which is of the form $y = x^{\frac{2}{3}} P\left(\frac{1}{x^{\frac{1}{3}}}\right)$, included in $y = x^{1-1/n} P\left(\frac{1}{x^{1/n}}\right)$, and the point is thus permissive; there is no prohibitive point, and the differential equation has a monotropic solution. This is, in fact, the rational solution $u = (z-c) - (z-c)^3$, or say $u = z - z^3$; the curve is thus given by the two equations $x = z - z^3$, $y = 1 - 3z^2$.

17. As a second example, take the differential equation

$$\left(\frac{du}{dz}\right)^3 - \left(\frac{du}{dz}\right)^2 + 4u^2 - 27u^4 = 0.$$

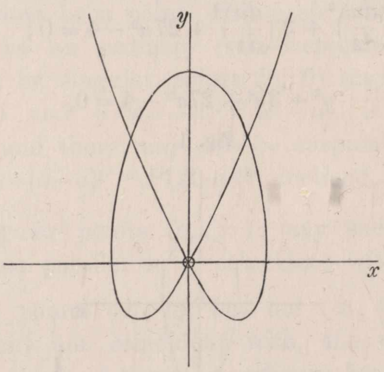
We have here the curve

$$y^3 - y^2 + 4x^2 - 27x^4 = 0,$$

which is a trinodal quartic curve, as shown in the figure. There is a node at the origin with the tangents $y^2 - 4x^2 = 0$, and, writing the equation in the form

$$(3y+1)(3y-2)^2 - (27x^2-2)^2 = 0,$$

Fig. 2.



we have for the other two nodes $x = \pm \frac{2}{3\sqrt{3}}$, $y = \frac{2}{3}$. The lowest points are given by $x = \pm \frac{2}{3\sqrt{3}}$, $y = -\frac{1}{3}$, viz. the line $y = -\frac{1}{3}$ is a horizontal tangent. Writing $x = 0$, we have $y^3 - y^2 = 0$, that is, $y^2 = 0$, the node at the origin, and $y = 1$, the height of the loop; and, writing $y=0$, we have $4x^2-27x^4=0$, that is, $x^2=0$, the node at the origin, and $x=\pm\frac{2}{3\sqrt{3}}$, the other two intersections with the axis of x. The tangents at these two points are vertical; as being on the axis, they are thus permissive points. And they are the *only* points with a vertical tangent; in fact, the point $(x=0,\ y=\infty)$ is a point of the curve, with the line ∞ as an osculating tangent (4-pointic intersection); hence the tangents from the point $x=0,\ y=\infty$ are the line ∞ counting four times, and the lines $x=\pm\frac{2}{3\sqrt{3}}$ touching at the points $\left(\pm\frac{2}{3\sqrt{3}},\ 0\right)$ as above. For the infinite branches, we have $y=3x+\ldots$, which is of the right form $y=x^{1+1/n}P\left(\frac{1}{x^{1/n}}\right)$. We thus see that there is no prohibitive point; and the differential equation has a monotropic solution accordingly.

Writing z in place of $z-c$, the solution in fact is

$$u=\frac{e^z-e^{-z}}{e^z+e^{-z}}-\left(\frac{e^z-e^{-z}}{e^z+e^{-z}}\right)^3;$$

hence, putting $\theta=\frac{e^z-e^{-z}}{e^z+e^{-z}}$, the curve is given by the two equations

$$x=\theta-\theta^3,\quad y=(1-3\theta^2)(1-\theta^2),\quad =1-4\theta^2+3\theta^4:$$

the monotropic function u satisfies the differential equation.

18. $F\left(u,\ \frac{du}{dz}\right)=0$ must be either a rational function, a singly periodic function, or a doubly periodic function of z; say the forms are

$$u=P(z),\quad u=P(e^{gz}),$$

and

$$u=P\{\operatorname{sn}(gz,\ k)\}+\operatorname{cn}(gz,\ k)\operatorname{dn}(gz,\ k)\,Q\{\operatorname{sn}(gz,\ k)\},$$

where P, Q denote rational functions. I do not at present consider the criteria (such as are given in Briot and Bouquet's Theorem V., p. 301) for determining by means of the curve which is the form of the integral. I remark, however, that in the first and second cases the curve is unicursal, while in the third case it is bicursal; or say that, according as the deficiency is $=0$ or 1, the integral is rational or simply periodic, or else it is doubly periodic. Moreover, the curve being unicursal, we can express the coordinates as equal to rational functions $P(\theta)$, $Q(\theta)$ of a parameter θ; and, being bicursal, we can express them as functions of the form

$$P(\theta)+P_1(\theta)\sqrt{1-\theta^2\,.\,1-k^2\theta^2},\quad Q(\theta)+Q_1(\theta)\sqrt{1-\theta^2\,.\,1-k^2\theta^2},$$

of the parameter θ; and supposing the coordinates $(x,\ y)$, that is, u and $\frac{du}{dz}$, thus expressed, it should be easy to establish the relation of θ with z, e^{gz} or $\operatorname{sn}(gz,\ k)$ in the three cases respectively.