859.

ON THE COMPLEX OF LINES WHICH MEET A UNICURSAL QUARTIC CURVE.

[From the *Proceedings of the London Mathematical Society*, vol. XVII. (1886), pp. 232—238.]

THE curve is taken to be that determined by the equations

$$x : y : z : w = 1 : \theta : \theta^3 : \theta^4,$$

viz. it is the common intersection of the quadric surface $\Theta = 0$, and the cubic surfaces $P = 0$, $Q = 0$, $R = 0$, where

$$\Theta = xw - yz, \quad P = x^2z - y^3, \quad Q = xz^2 - y^2w, \quad R = z^3 - yw^2.$$

Writing (a, b, c, f, g, h) as the six coordinates of a line, viz.

$$(a, b, c, f, g, h) = (\beta z - \gamma y,\ \gamma x - \alpha z,\ \alpha y - \beta x,\ \alpha w - \delta x,\ \beta w - \delta y,\ \gamma w - \delta z),$$

if $(\alpha, \beta, \gamma, \delta)$, (x, y, z, w) are the coordinates of any two points on the line; then, if the line meet the curve, we have

$$\begin{aligned} . \quad h\theta - g\theta^3 + a\theta^4 &= 0, \\ -h \quad . \quad + f\theta^3 + b\theta^4 &= 0, \\ g - f\theta \quad . \quad + c\theta^4 &= 0, \\ -a - b\theta - c\theta^3 \quad . \quad &= 0, \end{aligned}$$

from which four equations (equivalent, in virtue of the identity $af + bg + ch = 0$, to two independent equations), eliminating θ, we have the equation of the complex. The form may, of course, be modified at pleasure by means of the identity just referred to, but one form is

$$\Omega, = a^4 - b^3h + bf^2g + cg^3 - acfh + 2c^2h^2 - 4a^2ch + af^3 - a^3f = 0,$$

as may be verified by substituting therein the values $a = -b\theta - c\theta^3$, $g = f\theta - c\theta^4$, $h = f\theta^3 + b\theta^4$. The last-mentioned equation is thus the equation of the complex in question, in terms of the six coordinates (a, b, c, f, g, h).

If for the six coordinates we substitute their values, $\beta z - \gamma y$, &c., we obtain $\Omega, = (x, y, z, w)^4 (\alpha, \beta, \gamma, \delta)^4 = 0$, which, regarded as an equation in (x, y, z, w), is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, passing through the quartic curve; this equation should evidently be satisfied if only Θ, P, Q, R are each $=0$, viz. Ω must be a linear function of (Θ, P, Q, R); and by symmetry, it must be also a linear function of $(\Theta_0, P_0, Q_0, R_0)$, where

$$\Theta_0 = \alpha\delta - \beta\gamma, \quad P_0 = \alpha^2\gamma - \beta^3, \quad Q_0 = \alpha\gamma^2 - \beta^2\delta, \quad R_0 = \gamma^3 - \beta\delta^2,$$

viz. the form is $\Omega, = (\Theta, P, Q, R)(\Theta_0, P_0, Q_0, R_0)$, an expression with coefficients which are of the first or second degree in (x, y, z, w) and also of the first or second degree in $(\alpha, \beta, \gamma, \delta)$.

To work this out, I first arrange in powers and products of (α, δ), (β, γ), expressing the quartic functions of (x, y, z, w) in terms of (Θ, P, Q, R), as follows:

$\Omega =$

	a^4	$-b^3h$	$+bf^2g$	$+cg^3$	$-acfh$	$+2c^2h^2$	$-4a^2ch$	$+af^3$	$-a^3f$		
α^4										0	
$\alpha^3\delta$		$-z^4$	$+yzw^2$							$-z^4+yzw^2$	$-zR$
$\alpha^2\delta^2$			$-2xyzw$			$+2y^2z^2$				$-2xyzw+2y^2z^2$	$-2yz\Theta$
$\alpha\delta^3$			$+x^2yz$	$-y^4$						$+x^2yz-y^4$	$+yP$
δ^4										0	
$\alpha^3\beta$			$-zw^3$					$+zw^3$		0	
$\alpha^2\beta\delta$			$+2xzw^2$		$+yz^2w$			$-3xzw^2$		$-xzw^2+yz^2w$	$-zw\Theta$
$\alpha\beta\delta^2$			$-x^2zw$	$+3y^3w$	$-xyz^2$	$-4xyz^2$		$+3x^2zw$		$+2x^2zw+3y^3w-5xyz^2$	$+2xz\Theta-3yQ$
$\beta\delta^3$				$+xy^3$				$-x^3z$		$+xy^3-x^3z$	$-xP$
$\alpha^3\gamma$		$+z^3w$						$-yw^3$		$+z^3w-yw^3$	$+wR$
$\alpha^2\gamma\delta$		$+3xz^3$	$-xyw^2$		$-y^2zw$	$-4y^2zw$		$+3xyw^2$		$+3xz^3+2xyw^2-5y^2zw$	$+2yw\Theta+3zQ$
$\alpha\gamma\delta^2$			$+2x^2yw$		$+xyz^2$			$-3x^2yw$		$-x^2yw+xyz^2$	$-xy\Theta$
$\gamma\delta^3$				$-x^3y$				$+x^3y$		0	
$\alpha^2\beta^2$										0	
$\alpha^2\beta\gamma$			$+xw^3$		$-yzw^2$					$+xw^3-yzw^2$	$+w^2\Theta$
$\alpha^2\gamma^2$		$-3xz^2w$			$+y^2w^2$	$+2y^2w^2$				$-3xz^2w+3y^2w^2$	$-3wQ$
$\alpha\beta^2\delta$				$-3y^2w^2$	$-xz^2w$		$+4yz^3$			$-3y^2w^2-xz^2w+4yz^3$	$-4z^2\Theta+3wQ$
$\alpha\beta\gamma\delta$			$-2x^2w^2$		$+2xyzw$	$+8xyzw$	$-8y^2z^2$			$-2x^2w^2+10xyzw-8y^2z^2$	$+(-2xw+8yz)\Theta$
$\alpha\gamma^2\delta$		$-3x^2z^2$			$-xy^2w$		$+4y^3z$			$-3x^2z^2-xy^2w+4y^3z$	$-4y^2\Theta-3xQ$
$\beta^2\delta^2$				$-3xy^2w$	$+x^2z^2$	$+2x^2z^2$				$-3xy^2w+3x^2z^2$	$+3xQ$
$\beta\gamma\delta^2$			$+x^3w$		$-x^2yz$					$+x^3w-x^2yz$	$+x^2\Theta$
$\gamma^2\delta^2$										0	
$\alpha\beta^3$				$+yw^3$					$-z^3w$	$+yw^3-z^3w$	$-wR$
$\alpha\beta^2\gamma$					$+xzw^2$		$-4yz^2w$		$+3yz^2w$	$+xzw^2-yz^2w$	$+zw\Theta$
$\alpha\beta\gamma^2$					$-xyw^2$	$-4xyw^2$	$+8y^2zw$		$-3y^2zw$	$-5xyw^2+5y^2zw$	$-5yw\Theta$
$\alpha\gamma^3$		$+3x^2zw$					$-4y^3w$		$+y^3w$	$+3x^2zw-3y^3w$	$+3wP$
$\beta^3\delta$				$+3xyw^2$			$-4xz^3$		$+xz^3$	$+3xyw^2-3xz^3$	$-3xR$
$\beta^2\gamma\delta$					$-x^2zw$	$-4x^2zw$	$+8xyz^2$		$-3xyz^2$	$-5x^2zw+5xyz^2$	$-5xz\Theta$
$\beta\gamma^2\delta$					$+x^2yw$		$-4xy^2z$		$+3xy^2z$	$+x^2yw-xy^2z$	$+xy\Theta$
$\gamma^3\delta$		$+x^3z$							$-xy^3$	$+x^3z-xy^3$	$+xP$
β^4	$+z^4$			$-xw^3$						$+z^4-xw^3$	$+zR-w^2\Theta$
$\beta^3\gamma$	$-4yz^3$						$+4xz^2w$			$-4yz^3+4xz^2w$	$+4z^2\Theta$
$\beta^2\gamma^2$	$+6y^2z^2$					$+2x^2w^2$	$-8xyzw$			$+2x^2w^2-8xyzw+6y^2z^2$	$+(2xw-6yz)\Theta$
$\beta\gamma^3$	$-4y^3z$						$+4xy^2w$			$-4y^3z+4xy^2w$	$+4y^2\Theta$
γ^4	$+y^4$	$-x^3w$								$+y^4-x^3w$	$-yP-x^2\Theta$

Collecting the terms multiplied by P, Q, R, Θ, respectively, we have

$$\begin{aligned}\Omega = \ & P\{y\alpha\delta^3 - x\beta\delta^3 + 3w\alpha\gamma^3 + x\gamma^3\delta - y\gamma^4\} \\ & + Q\{-3y\alpha\beta\delta^2 + 3z\alpha^2\gamma\delta - 3w\alpha^2\gamma^2 + 3w\alpha\beta^2\delta - 3x\alpha\gamma^2\delta + 3x\beta^2\delta^2\} \\ & + R\{-z\alpha^3\delta + w\alpha^3\gamma - w\alpha\beta^3 - 3x\beta^3\delta + z\beta^4\} \\ & + \Theta\{-2yz\alpha^2\delta^2 - zw\alpha^2\beta\delta + 2xz\alpha\beta\delta^2 + 2yw\alpha^2\gamma\delta - xy\alpha\gamma\delta^2 \\ & \quad + w^2\alpha^2\beta\gamma - 4z^2\alpha\beta^2\delta + (-2xw + 8yz)\alpha\beta\gamma\delta - 4y^2\alpha\gamma^2 + x^2\beta\gamma\delta^2 \\ & \quad + zw\alpha\beta^2\gamma - 5yw\alpha\beta\gamma\delta - 5xz\beta^2\gamma\delta + xy\beta\gamma^2\delta \\ & \quad - w^2\beta^4 + 4z^2\beta^3\gamma + (2xw - 6yz)\beta^2\gamma^2 + 4y^2\beta\gamma^3 - x^2\gamma^4\},\end{aligned}$$

which may be written as follows:—

$$\begin{aligned}\Omega = \ & P\{y(\alpha\delta^3 - \gamma^4) + x(\gamma^3\delta - \beta\delta^3)\} & + P(3w\alpha\gamma^3) \\ & + Q\{3x(\beta^2\delta^2 - \alpha\gamma^2\delta) + 3w(\alpha\beta^2\delta - \alpha^2\gamma^2)\} & + Q(3z\alpha^2\gamma\delta - 3y\alpha\beta\delta^2) \\ & + R\{-z(\alpha^3\delta - \beta^4) + w(\alpha^3\gamma - \alpha\beta^3)\} & + R(-3x\beta^3\delta) \\ & + \Theta\{zw(-\alpha^2\beta\delta + \alpha\beta^2\gamma) & \\ & \quad + xz\,2(\alpha\beta\delta^2 - \beta^2\gamma\delta) & + \Theta(-3xz\beta^2\gamma\delta) \\ & \quad + yw\,2(\alpha^2\gamma\delta - \alpha\beta\gamma^2) & + \Theta(-3yw\alpha\beta\gamma^2) \\ & \quad + xy(-\alpha\gamma\delta^2 + \beta\gamma^2\delta) & \\ & \quad + xw\,2(-\alpha\beta\gamma\delta + \beta^2\gamma^2) & \\ & \quad + yz(-2\alpha^2\delta^2 + 8\alpha\beta\gamma\delta - 6\beta^2\gamma^2) & \\ & \quad + x^2(\beta\gamma\delta^2 - \gamma^4) & \\ & \quad + y^2\,4(-\alpha\gamma^2\delta + \beta\gamma^3) & \\ & \quad + z^2\,4(-\alpha\beta^2\delta + \beta^3\gamma) & \\ & \quad + w^2(\alpha^2\beta\gamma - \beta^4) \qquad\qquad \}, & \end{aligned}$$

in which all the terms contained in the $\{\ \}$ admit of expression in terms of P_0, Q_0, R_0, Θ_0; the remaining six terms not included within $\{\ \}$ may be written

$$3wP\alpha(\gamma^3 - \beta\delta^2) + 3(wP - yQ)\alpha\beta\delta^2 - 3\Theta xz\beta^2\gamma\delta,$$

$$-3xR\delta(\beta^3 - \alpha^2\gamma) + 3(-xR + zQ)\alpha^2\gamma\delta - 3\Theta yw\alpha\beta\gamma^2;$$

which, observing that $wP - yQ = xz\Theta$, and $-xR + zQ = yw\Theta$, are

$$-3wP\alpha(\gamma^3 - \beta\delta^2) + 3xz\Theta(\alpha\beta\delta^2 - \beta^2\gamma\delta),$$

$$-3xR\delta(\beta^3 - \alpha^2\gamma) + 3yw\Theta(\alpha^2\gamma\delta - \alpha\beta\gamma^2).$$

The expression thus becomes

$$\begin{aligned}
\Omega = \; P\,. \quad & x(\gamma^3\delta - \beta\delta^3) && = x\delta R_0 \\
& + y(\alpha\delta^3 - \gamma^4) && = y(-\gamma R_0 + \delta^2\Theta) \\
& + 3w\alpha(\gamma^3 - \beta\delta^2) && = 3w\alpha R_0 \\
+ Q\,. & - 3x(\beta^2\delta^2 - \alpha\gamma^2\delta) && = - 3x\delta Q_0 \\
& + 3w(\alpha\beta^2\delta - \alpha^2\gamma^2) && = - 3w\alpha Q_0 \\
+ R\,. & - 3x\delta(\beta^3 - \alpha^2\gamma) && = 3x\delta P_0 \\
& - z(\alpha^3\delta - \beta^4) && = z(-\beta P_0 - \alpha^2\Theta_0) \\
& + w(\alpha^3\gamma - \alpha\beta^3) && = w\alpha P_0 \\
+ \Theta\,. \quad & zw(-\alpha^2\beta\delta + \alpha\beta^2\gamma) && = - zw\alpha\beta\Theta_0 \\
& + 5xz(\alpha\beta\delta^2 - \beta^2\gamma\delta) && = 5xz\beta\delta\Theta_0 \\
& + 5yw(\alpha^2\gamma\delta - \alpha\beta\gamma^2) && = 5yw\alpha\gamma\Theta_0 \\
& + xy(-\alpha\gamma\delta^2 + \beta\gamma^2\delta) && = - xy\gamma\delta\Theta_0 \\
& + 2xw(-\alpha\beta\gamma\delta + \beta^2\gamma^2) && = - 2xw\beta\gamma\Theta_0 \\
& + yz(-2\alpha^2\delta^2 + 8\alpha\beta\gamma\delta - 6\beta^2\gamma^2) && = - 2yz(\alpha\delta - 3\beta\gamma)\Theta_0 \\
& + x^2(\beta\gamma\delta^2 - \gamma^4) && = - x^2\gamma R_0 \\
& + 4y^2(-\alpha\gamma^2\delta + \beta\gamma^3) && = - 4y^2\gamma^2\Theta_0 \\
& + 4z^2(-\alpha\beta^2\delta + \beta^3\gamma) && = - 4z^2\beta^2\Theta_0 \\
& + w^2(\alpha^2\beta\gamma - \beta^4) && = w^2\beta P\,;
\end{aligned}$$

and we thus finally obtain

$$\begin{aligned}
\Omega = \; & PR_0(3\alpha w - \gamma y + \delta x) \\
& + RP_0(3\delta x - \beta z + \alpha w) \\
& + P\Theta_0\,.\,\delta^2 y \\
& + R\Theta_0\,.\,-\alpha^2 z \\
& + P_0\Theta\,.\,\beta w^2 \\
& + R_0\Theta\,.\,-\gamma x^2 \\
& - QQ_0\,.\,-3(\alpha w + \delta x) \\
& + \Theta\Theta_0\{-\alpha\beta zw - \gamma\delta xy + 5\beta\delta xz + 5\alpha\gamma yw - 2\beta\gamma xw - 2\alpha\delta yz \\
& \qquad - 4\gamma^2 y^2 + 6\beta\gamma yz - 4\beta^2 z^2\},
\end{aligned}$$

viz. $\Omega = 0$ is the equation of the cone, vertex $(\alpha, \beta, \gamma, \delta)$, which passes through the quartic curve $x : y : z : w = 1 : \theta : \theta^3 : \theta^4$. As regards the symmetry of this expression, it is to be remarked that, changing (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$ into (w, z, y, x) and $(\delta, \gamma, \beta, \alpha)$ respectively, we change (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$ into $(\Theta, -R, -Q, -P)$ and $(\Theta_0, -R_0, -Q_0, -P_0)$, respectively, and so leave Ω unaltered. Again, interchanging (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$, we interchange (Θ, P, Q, R) and $(\Theta_0, P_0, Q_0, R_0)$, and so leave Ω unaltered.