856.

NOTE ON A CUBIC EQUATION.

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CONSIDER the cubic equation

$$x^3 + 3cx + d = 0;$$

then effecting upon this the Tschirnhausen-Hermite transformation

$$y = xT_1 + (x^2 + 2c)\,T_2,$$

the resulting equation in y is

$$\begin{aligned} &y^3 + 3y\,(cT_1^2 + dT_1T_2 - c^2T_2^2) \\ &\quad + dT_1^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3 = 0, \end{aligned}$$

and this will be

$$y^3 + 3cy + d = 0,$$

if only

$$\begin{aligned} c &= cT_1^2 + dT_1T_2 - c^2T_2^2, \\ d &= dT_1^3 - 6c^2T_1^2T_2 - 3cdT_1T_2^2 - (d^2 + 2c^3)\,T_2^3, \end{aligned}$$

equations which give

$$(d^2 + 4c^3) = (d^2 + 4c^3)\,(T_1^3 + 3cT_1T_2^2 + dT_2^3)^2,$$

viz. assuming that $d^2 + 4c^3$ not $= 0$, this is

$$1 = T_1^3 + 3cT_1T_2^2 + dT_2^3.$$

Hence the coefficients T_1, T_2 being such as to satisfy these relations, the equation in z is identical with the equation in x; or, what is the same thing, if α, β, γ are the roots of the equation in x, then we have between these roots the relations

$$\begin{aligned} \beta &= \alpha T_1 + (\alpha^2 + 2c)\,T_2, \\ \gamma &= \beta T_1 + (\beta^2 + 2c)\,T_2, \\ \alpha &= \gamma T_1 + (\gamma^2 + 2c)\,T_2, \end{aligned}$$

viz. the general cubic equation $x^3+3cx+d=0$, adjoining thereto the radicals T_1, T_2 may be regarded as an Abelian equation.

In particular, if $c, d=-1, 1$, then we may write $T_1=0$, $T_2=1$; the cubic equation is here

$$x^3+3x-1=0,$$

and the roots α, β, γ are such that $\beta=\alpha^2-2$, $\gamma=\beta^2-2$, $\alpha=\gamma^2-2$; in fact, taking θ a primitive ninth root of unity, $\theta^6+\theta^3+1=0$; we have $\alpha, \beta, \gamma=\theta+\theta^8, \theta^2+\theta^7, \theta^4+\theta^5$; values which satisfy $x^3+3x-1=0$, and the relations in question.

The same question may be considered from a different point of view. Take the transforming equation to be

$$y=A+Bx+Cx^2,$$

then assuming that the values of y corresponding to the values $x=\alpha, \beta, \gamma$ are β, γ, α respectively, we have

$$\beta=A+B\alpha+C\alpha^2,$$

$$\gamma=A+B\beta+C\beta^2,$$

$$\alpha=A+B\gamma+C\gamma^2,$$

and the transforming equation thus is

$$\begin{vmatrix} y, & 1, & x, & x^2 \\ \beta, & 1, & \alpha, & \alpha^2 \\ \gamma, & 1, & \beta, & \beta^2 \\ \alpha, & 1, & \gamma, & \gamma^2 \end{vmatrix}=0.$$

This may also be written

$$\begin{aligned} &(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\{y-\tfrac{1}{2}(\alpha+\beta+\gamma+x)\} \\ = \quad &\beta^2\gamma^2+\gamma^2\alpha^2+\alpha^2\beta^2-\tfrac{1}{2}(\beta^3\gamma+\beta\gamma^3+\gamma^3\alpha+\gamma\alpha^3+\alpha^3\beta+\alpha\beta^3) \\ &+x\{\alpha^3+\beta^3+\gamma^3-\tfrac{1}{2}(\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2+\alpha^2\beta+\alpha\beta^2)\} \\ &+x^2\{\beta\gamma+\gamma\alpha+\alpha\beta-(\alpha^2+\beta^2+\gamma^2)\}. \end{aligned}$$

We have

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=\frac{-27}{a^4}(a^2d^2+4ac^3+4b^3d-3b^2c^2-6abcd),$$

$$=\frac{-27}{a^4}\Delta,$$

or, say

$$(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)=\frac{3(\omega-\omega^2)}{a^2}\sqrt{(\Delta)},$$

if Δ be the discriminant, and ω an imaginary cube root of unity, $\{(\omega-\omega^2)^2=-3\}$.

The remaining functions of α, β, γ are of course expressible rationally in terms of the coefficients: we have

$$\begin{aligned}
\Sigma\beta^2\gamma^2 &= \frac{1}{a^2}(-6bd+9c^2),\\
\Sigma\beta^3\gamma &= \frac{1}{a^3}(-3abd-18ac^2+27b^2c),\\
\Sigma\alpha^3 &= \frac{1}{a^3}(-3a^2d+27abc-27b^3),\\
\Sigma\beta^2\gamma &= \frac{1}{a^2}(3ad-9bc),\\
\Sigma\beta\gamma &= \frac{3c}{a},\\
\Sigma\alpha^2 &= \frac{1}{a^2}(9b^2-6ac),
\end{aligned}$$

and the final result is

$$\begin{aligned}
\tfrac{1}{3}(\omega-\omega^2)\sqrt{(\Delta)}\{2a(y+x)+3b\} = {} & -abd+4ac^2-3b^2c\\
& +x\,(-a^2d+7abc-6b^3)\\
& +x^2(\;2a^2c-2ab^2);
\end{aligned}$$

viz. we have thus an automorphic transformation of the equation

$$ax^3+3bx^2+3cx+d=0.$$