855.

SOLUTION OF $(a, b, c, d) = (a^2, b^2, c^2, d^2)$.

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It is required to find four quantities (no one of them zero) which are in some order or other equal to their squares, say

$$(a, b, c, d) = (a^2, b^2, c^2, d^2).$$

Supposing that the required quantities (a, b, c, d) are the roots of the biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

(s not $= 0$), then the function

$$(x^4 + qx^2 + s)^2 - (px^3 + rx)^2 \text{ must be } = x^8 + px^6 + qx^4 + rx^2 + s;$$

and we have thus the conditions

$$2q - p^2 = p, \quad 2s + q^2 - 2pr = q, \quad 2qs - r^2 = r, \quad s^2 = s,$$

the last of which (since s is not $= 0$) gives $s = 1$, and the others then become

$$2q = p^2 + p, \quad 2(pr - 1) = q^2 - q, \quad 2q = r^2 + r;$$

viz. regarding p, q, r as the coordinates of a point in space, this is determined as the intersection of three quadric surfaces, and the number of solutions is thus $= 8$.

We, in fact, have $2q = p^2 + p = r^2 + r$; that is, $p^2 + p = r^2 + r$, or $(p - r)(p + r + 1) = 0$; hence $r = p$ or $r = -1 - p$.

First, if $r = p$; here $2q = p^2 + p$, $2(p^2 - 1) = q^2 - q$: the last equation multiplied by 4 gives

$$8(p^2 - 1) = (p^2 + p)(p^2 + p - 2), \ = p(p^2 - 1)(p + 2),$$

that is, $p^2 - 1 = 0$ or $p^2 + 2p - 8 = 0$.

If $p^2 - 1 = 0$, then either $p = 1$, giving $q = 1$, $r = 1$, and hence the equation is $x^4 + x^3 + x^2 + x + 1 = 0$; or else $p = -1$, giving $q = 0$, $r = -1$, and hence the equation is $x^4 - x^3 - x + 1 = 0$, that is, $(x-1)^2(x^2 + x + 1) = 0$.

If $p^2 + 2p - 8 = 0$, then either $p = 2$, giving $q = 3$, $r = 2$, and hence the equation is $x^4 + 2x^3 + 3x^2 + 2x + 1$, that is,

$$(x^2 + x + 1)^2 = 0;$$

or else $p = -4$, giving $q = 6$, $r = -4$, and hence the equation is $x^4 - 4x^3 + 6x^2 - 4x + 1 = 0$, that is, $(x-1)^4 = 0$.

Secondly, if $r = -1 - p$; here

$$2q = p^2 + p, \quad 2(-p^2 - p - 1) = q^2 - q;$$

the last equation multiplied by 4 gives

$$8(-p^2 - p - 1) = (p^2 + p)(p^2 + p - 2),$$

that is,

$$p^4 + 2p^3 + 7p^2 + 6p + 8 = 0, \text{ or } (p^2 + p + 4)(p^2 + p + 2) = 0.$$

If $p^2 + p + 4 = 0$, then $p = \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}$, whence

$$r = \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}, \quad 2q = p^2 + p, \ = -4, \text{ or } q = -2;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x^3 - 2x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(15)}\}x + 1 = 0.$$

If $p^2 + p + 2 = 0$, then $p = \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}$; whence

$$r = \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}, \quad 2q = p^2 + p, \ = -2, \text{ or } q = -1;$$

and the equation is

$$x^4 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x^3 - x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x + 1 = 0,$$

that is,

$$(x-1)[x^3 + \tfrac{1}{2}\{1 \pm i\sqrt{(7)}\}x^2 + \tfrac{1}{2}\{-1 \pm i\sqrt{(7)}\}x - 1] = 0.$$

We thus see that the eight equations are

1 $(x-1)^4 = 0$,

1 $(x^2 + x + 1)^2 = 0$,

1 $(x-1)^2(x^2 + x + 1) = 0$,

1 $x^4 + x^3 + x^2 + x + 1 = 0$,

2 $(x-1)\{x^3 + \tfrac{1}{2}(1 \pm i\sqrt{7})x^2 + \tfrac{1}{2}(-1 \pm i\sqrt{7})x - 1\} = 0$,

2 $x^4 + \tfrac{1}{2}(-1 \pm i\sqrt{15})x^3 - 2x^2 + \tfrac{1}{2}(-1 \mp i\sqrt{15})x + 1 = 0$,

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and it hence appears that, writing γ, ϵ, θ to denote respectively an imaginary cube root, fifth root, and seventh root of unity, then the values of (a, b, c, d) are

$$\begin{array}{cccc} 1, & 1, & 1, & 1\,; \\ \gamma, & \gamma, & \gamma^2, & \gamma^2; \\ 1, & 1, & \gamma, & \gamma^2; \\ \epsilon, & \epsilon^2, & \epsilon^3, & \epsilon^4; \\ \epsilon\gamma, & \epsilon^3\gamma^2, & \epsilon^4\gamma, & \epsilon^3\gamma^2; \\ \epsilon^2\gamma, & \epsilon^4\gamma^2, & \epsilon^3\gamma, & \epsilon\gamma^2; \\ 1, & \theta, & \theta^2, & \theta^4; \\ 1, & \theta^3, & \theta^6, & \theta^5; \end{array}$$

viz. for each of these systems we have the required relation

$$(a, b, c, d)=(a^2, b^2, c^2, d^2).$$

It may be noticed that out of the eight equations we have the following three which are irreducible:—

$$x^4+x^3+x^2+x+1=0,$$
$$x^4+\tfrac{1}{2}(-1+i\sqrt{15})\,x^3-2x^2+\tfrac{1}{2}(-1-i\sqrt{15})\,x+1=0,$$
$$x^4+\tfrac{1}{2}(-1-i\sqrt{15})\,x^3-2x^2+\tfrac{1}{2}(-1+i\sqrt{15})\,x+1=0.$$

Each of these is an Abelian equation, viz. the roots are of the form

$$a,\ \theta(a),\ \theta^2(a),\ \theta^3(a),\ (=a,\ a^2,\ a^4,\ a^8),$$

where $\theta^4(a)=a$, not identically but in virtue of the value of a, viz. we have $\theta^4(a)=a^{16}=a$, in virtue of $a^{15}=1$: (in the first equation $a^5=1$, and therefore $a^{15}=1$; in each of the other two, a^{15} is the lowest power which is $=1$).

In the first equation, we have evidently

$$x^4+x^3+x^2+x+1$$

as the irreducible factor of x^5-1.

The second and third equations combined together give

$$(x^4-\tfrac{1}{2}x^3-2x^2-\tfrac{1}{2}x+1)^2+\tfrac{15}{4}(x^3-x)^2=0\,;$$

that is,

$$x^8-x^7+x^5-x^4+x^3-x+1=0,$$

where the left-hand side is the irreducible factor of $x^{15}-1$.