854.

AN ALGEBRAICAL TRANSFORMATION.

[From the *Messenger of Mathematics*, vol. XV. (1886), pp. 58, 59.]

THE following algebraical transformation occurs in a paper by Hermite "On the theory of the Modular Equations," *Comptes Rendus*, t. XLVIII. (1859), p. 1100.

Writing $q = 1 - 2u^8$, $l = 1 - 2v^8$, then in the transformation of the fifth order, the modular equation was expressed by Jacobi in the form

$$\Omega, \ = (q-l)^6 - 256(1-q^2)(1-l^2)\{16ql(9-ql)^2 + 9(45-ql)(q-l)^2\}, \ = 0;$$

and if we write herein $q = 1 - 2x$, $l = \dfrac{x+1}{x-1}$, or, what is the same thing, establish between q, l the relation $q - l = 3 + ql$, that is, between u, v the relation $v^8 = 1 \div (1 - u^8)$, then the function Ω becomes

$$\Omega = \frac{64}{(1-x)^6}\{(x^2-x+1)^3 + 2^7(x^2-x)^2\}\{(x^2-x+1)^3 + 2^7 . 3^3(x^2-x)^2\};$$

or, what is the same thing, the equation $\Omega = 0$ gives for $\dfrac{(x^2-x+1)^3}{(x^2-x)^2}$ the values -2^7 and $-2^7 . 3^3$.

We, in fact, have

$$q - l \ = 3 + ql = \frac{2(x^2-x+1)}{1-x},$$

$$1 - q^2 = 4x(1-x), \quad 1 - l^2 = \frac{-4x}{(1-x)^2},$$

and therefore

$$(1-q^2)(1-l^2) = \frac{-16x^2}{1-x}.$$

Hence

$$\Omega = \frac{64}{(1-x)^6}[(x^2-x+1)^6 + 64(1-x)^5 \times x^2 \{16ql(9-ql)^2 + 9(45-ql)(3+ql)^2\}];$$

and, putting for a moment $ql = \theta - 3$, the term in $\{\ \}$ is found to be

$$= 7\theta^3 + 3456\theta - 6912;$$

viz. this is

$$= \frac{56(x^2-x+1)^2}{(1-x)^3} + \frac{6912(x^2-x+1)}{1-x} - 6912,$$

$$= \frac{8}{(1-x)^3}\{7(x^2-x+1)^3 + 864(x-1)^2(x^2-x+1) + 864(x-1)^3\},$$

$$= \frac{8}{(1-x)^3}\{7(x^2-x+1)^3 + 864(x^2-x)^2\}.$$

Hence

$$\Omega = \frac{64}{(1-x)^6}[(x^2-x+1)^6 + 512(x^2-x)^2\{7(x^2-x+1)^3 + 864(x^2-x)^2\}],$$

which is

$$= \frac{64}{(1-x)^6}\{(x^2-x+1)^3 + 2^7(x^2-x)^2\}\{(x^2-x+1)^3 + 2^7 . 3^3(x^2-x)^2\}.$$