851.

ON LINEAR DIFFERENTIAL EQUATIONS (THE THEORY OF DECOMPOSITION).

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1. In the theory of linear differential equations the question arises, to decompose a quantic $\left(*\between\frac{d}{dx}, 1\right)^n$ into linear factors: we have, for instance, a differential equation

$$\frac{d^2y}{dx^2}+p\frac{dy}{dx}+qy=0,$$

which is to be expressed in the form

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right)y=0,$$

or say we have to express $\frac{d^2}{dx^2}+p\frac{d}{dx}+q$ as a product of linear factors

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right).$$

The problem is analogous to, but wholly distinct from, that of the resolution of an algebraic equation: using accents to denote differentiation in regard to x, the relation (in the simple case just referred to) between the coefficients (p, q) and the roots (α, β) is (not $p=\alpha+\beta$, $q=\alpha\beta$, but in place thereof) $p=\alpha+\beta$, $q=\alpha\beta+\beta'$; and it thus appears that there is the important distinction that, in the present problem, the order of the factors is not indifferent.

2. The problem may be solved when the general solution of the differential equation is known, or, what is the same thing, by means of n particular solutions

of the equation; it is not here considered in this point of view, but the intention is to treat it directly by means of the relations between the coefficients and the roots: thus in the above case, $p=\alpha+\beta$, $q=\alpha\beta+\beta'$, we may (1) find first β and then α, viz. eliminating α we have $\beta'+\beta(p-\beta)-q=0$, β determined by a differential equation (not linear) of the first order; and then $\alpha=p-\beta$; or we may (2) find first α and then β, viz. eliminating β, we have $\alpha'-p'+\alpha(\alpha-p)+q=0$, α determined by a differential equation (not linear) of the first order, and then $\beta=p-\alpha$.

In the case of a product

$$\left(\frac{d}{dx}+\alpha\right)\left(\frac{d}{dx}+\beta\right)\left(\frac{d}{dx}+\gamma\right)\cdots$$

of more than two factors, the roots might be determined in any order; but the two orders which naturally present themselves and which will be alone considered are, say, the reverse order $(\ldots, \gamma, \beta, \alpha)$ and the direct order $(\alpha, \beta, \gamma, \ldots)$.

3. *The Reverse Order.* Writing D for $\dfrac{d}{dx}$, we assume

$$\begin{aligned} D+\alpha &= (1,\ p_1 \between D,\ 1), \\ (D+\alpha)(D+\beta) &= (1,\ p_2,\ q_2 \between D,\ 1)^2, \\ (D+\alpha)(D+\beta)(D+\gamma) &= (1,\ p_3,\ q_3,\ r_3 \between D,\ 1)^3, \end{aligned}$$

and so on. Then using accents to denote differentiation, we find

$$\begin{aligned}
&(1)\quad p_1 = \alpha. \\
&(2)\quad p_2 = \beta + p_1, \\
&\qquad\ q_2 = \beta' + p_1\beta. \\
&(3)\quad p_3 = \gamma + p_2, \\
&\qquad\ q_3 = 2\gamma' + p_2\gamma + q_2, \\
&\qquad\ r_3 = \gamma'' + p_2\gamma' + q_2\gamma. \\
&(4)\quad p_4 = \delta + p_3, \\
&\qquad\ q_4 = 3\delta' + p_3\delta + q_3, \\
&\qquad\ r_4 = 3\delta'' + 2p_3\delta' + q_3\delta + r_3, \\
&\qquad\ s_4 = \delta''' + p_3\delta'' + q_3\delta' + r_3\delta,
\end{aligned}$$

where the law is obvious: thus in the last set of equations, the several columns (after the first) contain the factors p_3, q_3, r_3, s_3 respectively; and there is in each column a head term with the coefficient unity: omitting these head terms, we have in the several columns the sets $(\delta, 3\delta', 3\delta'', \delta''')$, $(\delta, 2\delta', \delta'')$, (δ, δ'), δ, of derivatives of the root δ.

4. We may from each set of equations determine in order the coefficients of lower rank contained in the equations, and the last equation of each set then gives an equation independent of these coefficients of lower rank. Thus set (2) gives

$$\begin{aligned} p_1 &= -\beta & 0 &= \beta' - \beta^2 \\ &\quad +p_2; & &\quad +p_2\beta \\ & & &\quad -p_2. \end{aligned}$$

Set (3) gives

$$\begin{array}{lll} p_2 = -\gamma & q_2 = -2\gamma' + \gamma^2 & 0 = \gamma'' - 3\gamma\gamma' + \gamma^3 \\ \quad + p_3; & \quad - p_3\gamma & \quad + p_3(\gamma' - \gamma^2) \\ & \quad + q_3; & \quad + q_3\gamma \\ & & \quad - r_3. \end{array}$$

Set (4) gives

$$\begin{array}{lll} p_3 = -\delta & q_3 = -3\delta' + \delta^2 & r_3 = -3\delta'' + 5\delta\delta' - \delta^3 \\ \quad + p_4; & \quad - p_4\delta & \quad + p_4(-2\delta' + \delta^2) \\ & \quad + q_4; & \quad - q_4\delta \\ & & \quad + r_4; \end{array}$$

$$\begin{aligned} 0 = {} & \delta''' - 4\delta\delta'' + 6\delta^2\delta' - 3\delta'^2 - \delta^4 \\ & + p_4(\delta'' - 3\delta\delta' + \delta^3) \\ & + q_4(\delta' - \delta^2) \\ & + r_4\delta \\ & - s_4; \end{aligned}$$

and so on.

5. Thus suppose (p_4, q_4, r_4, s_4) are given, we have δ determined by a differential equation (not linear) of the third order; and δ being known, p_3, q_3, r_3 are also known; then γ is determined by a differential equation (not linear) of the second order, and γ being known, p_2, q_2 are also known; then β is determined by a differential equation (not linear) of the first order, and β being known, p_1, that is, α is also known.

Comparing the last equations of each set, that is, the equations for the determination of β, γ, δ, ... respectively, it will be observed that they depend on the single series of derivatives

$$\begin{aligned} & -1, \\ & \phi, \\ & \phi' - \phi^2, \\ & \phi'' - 3\phi\phi' + \phi^3, \\ & \phi''' - 4\phi\phi'' + 6\phi^2\phi' - 3\phi'^2 - \phi^4, \end{aligned}$$

where, calling the successive terms A, B, C, D, E, ..., we have

$$B = A' - \phi A,\ C = B' - \phi B,\ D = C' - \phi C,\ E = D' - \phi D.$$

6. *Direct Order*. Considering the product

$$\left(\frac{d}{dx} + \alpha\right)\left(\frac{d}{dx} + \beta\right)\left(\frac{d}{dx} + \gamma\right)\left(\frac{d}{dx} + \delta\right)$$

of four factors, and assuming

$$\begin{aligned} D+\delta &= (1,\ p_1 \between D,\ 1), \\ (D+\gamma)(D+\delta) &= (1,\ p_2,\ q_2 \between D,\ 1)^2, \\ (D+\beta)(D+\gamma)(D+\delta) &= (1,\ p_3,\ q_3,\ r_3 \between D,\ 1)^3, \\ (D+\alpha)(D+\beta)(D+\gamma)(D+\delta) &= (1,\ p_4,\ q_4,\ r_4,\ s_4 \between D,\ 1)^4, \end{aligned}$$

we have the sets of equations

$$\begin{aligned} &(1)\quad p_1 = \delta. \\ &(2)\quad p_2 = \gamma + p_1, \\ &\qquad\ q_2 = \quad \gamma p_1 + p_1'. \\ &(3)\quad p_3 = \beta + p_2, \\ &\qquad\ q_3 = \quad \beta p_2 + p_2' + q_2, \\ &\qquad\ r_3 = \qquad\qquad \beta q_2 + q_2'. \\ &(4)\quad p_4 = \alpha + p_3, \\ &\qquad\ q_4 = \quad \alpha p_3 + p_3' + q_3, \\ &\qquad\ r_4 = \qquad\qquad \alpha q_3 + q_3' + r_3, \\ &\qquad\ s_4 = \qquad\qquad\qquad \alpha r_3 + r_3'. \end{aligned}$$

7. I stop to remark that there would have been a convenience in considering the product

$$\cdots\left(\frac{d}{dx}+\delta\right)\left(\frac{d}{dx}+\gamma\right)\left(\frac{d}{dx}+\beta\right)\left(\frac{d}{dx}+\alpha\right);$$

for the first, second, third, &c., sets of equations would then have contained α, β, γ, &c., respectively; but for better comparison with the equations of the reverse order, I have preferred not to make this change of notation.

8. The set (1) gives

$$\delta - p_1 = 0.$$

Set (2) gives

$$p_1 = -\gamma + p_2, \quad 0 = \begin{array}{l} \gamma' + \gamma^2 \\ -p_2\gamma - p_2' \\ +q_2. \end{array}$$

Set (3) gives

$$p_2 = -\beta + p_3, \quad q_2 = \begin{array}{l} \beta' + \beta^2 \\ -p_3\beta - p_3' \\ +q_3, \end{array} \quad 0 = \begin{array}{l} \beta'' + 3\beta\beta' + \beta^3 \\ -p_3(\beta'+\beta^2) - 2p\beta_3 - p_3'' \\ +q_3\beta + q_3' \\ -r_3. \end{array}$$

Set (4) gives

$$p_3 = -\alpha + p_4, \quad q_3 = \begin{array}{l} \alpha' + \alpha^2 \\ -p_4\alpha - p' \\ +p_4, \end{array} \quad r_3 = \begin{array}{l} -(\alpha'' + 3\alpha\alpha' + \alpha^3) \\ +p_4(\alpha^2 + \alpha') + 2p_4'\alpha + p_4'' \\ -q_4\alpha - q_4' \\ +r_4, \end{array}$$

$$\begin{aligned} 0 = \ & \alpha''' + 4\alpha\alpha'' + 6\alpha^2\alpha' + 3\alpha'^2 + \alpha^4 \\ & - p_4(\alpha'' + 3\alpha\alpha' + \alpha^3) - p_4'(3\alpha' + 3\alpha^2) - 3p_4''\alpha - p_4''' \\ & + q_4(\alpha' + \alpha^2) + 2q_4'\alpha + q_4'' \\ & - r_4\alpha - r_4' \\ & + s_4, \end{aligned}$$

which differ in form from the equations belonging to the reverse order in containing the derived functions p_4', p_4'', p_4''', q_4', ... of the coefficients.

9. Taking p_4, q_4, r_4, s_4 as known, we have α determined by a differential equation (not linear) of the third order; and α being known, we know p_3, q_3, r_3. We then have β determined by a differential equation (not linear) of the second order; and β being known, we know p_2, q_2. We then have γ determined by a differential equation (not linear) of the first order; and γ being known, we know p_1, that is, δ.