850.

ON LINEAR DIFFERENTIAL EQUATIONS.

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1. THE researches of Fuchs, Thomé, Frobenius, Tannery, Floquet, and others, relate to linear differential equations of the form

$$p_0 \frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y = 0,$$

or say

$$(p_0, p_1, \ldots, p_m \between \frac{d}{dx}, 1)^m y = 0,$$

where $p_0, p_1, \ldots, p_m$ are rational and integral functions of the independent variable x; any common factor of all the functions could of course be thrown out, and it is therefore assumed that the functions have no common factor. It is to be throughout understood that x and y denote complex magnitudes, which may be regarded as points in the infinite plane; viz. $x, = \xi + i\eta$, is the point the coordinates whereof are ξ, η: and similarly for y.

2. Suppose $x - a$ is *not* a factor of p_0, the point $x = a$ is in this case said to be an ordinary point in regard to the differential equation; and let $y_0, y_1, y_2, \ldots, y_{m-1}$ be arbitrary constants denoting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{m-1}y}{dx^{m-1}}$ for the point $x = a$. We can from the differential equation, and the equations derived therefrom by successive differentiations in regard to x, obtain the values for $x = a$ of the subsequent differential coefficients $\frac{d^m y}{dx^m}, \frac{d^{m+1}y}{dx^{m+1}}, \ldots$, say these are $y_m, y_{m+1}, \ldots$; viz. the value of each of these quantities will be determined as a linear function of $y_0, y_1, \ldots, y_{m-1}$; and we thus have a development of y in positive integer powers of $x - a$, viz. this is

$$y = y_0 + y_1(x-a) + \frac{1}{1 \cdot 2} y_2 (x-a)^2 + \ldots,$$

which will in fact be a sum of m determinate series multiplied by $y_0, y_1, y_2, \ldots, y_{m-1}$ respectively; say the form is

$$y = y_0 X_0 + y_1 X_1 + \ldots + y_{m-1} X_{m-1},$$

where $X_0, X_1, \ldots, X_{m-1}$ are each of them a series of positive integer powers of $x-a$. Each of these series will be convergent for values of x sufficiently near to a, or say for points x within the domain of the point a; and since $y_0, y_1, \ldots, y_{m-1}$ are arbitrary, each series separately will satisfy the differential equation; and we have thus m particular integrals of the differential equation.

3. Suppose next that $x-a$ is a factor of p_0, the point $x=a$ is in this case said to be a singular point in regard to the differential equation. The foregoing process of development fails, as leading to infinite values of $\frac{d^m y}{dx^m}, \frac{d^{m+1} y}{dx^{m+1}}, \ldots$; and we have to consider the developments of y which belong to the neighbourhood of the point $x=a$, or say to the domain of the singular point $x=a$. This has to be done separately in regard to each of the singular points, and among these we have, it may be, the point ∞. To decide whether ∞ is or is not a singular point, we may in the differential equation write $x=1/t$; and then transforming to this new variable, and throwing out any common factor, the coefficients of the transformed equation will be rational and integral functions of t, without any common factor; say these are $P_0, P_1, \ldots, P_m$; if t is not a factor of P_0, then ∞ will be an ordinary point of the original equation; but if t is a factor of P_0, then ∞ will be a singular point of the original equation.

4. In considering the singular point $x=a$, we may, it is clear, transform the equation to this point as origin, and it is convenient to do this; supposing it done, we have an equation wherein $x=0$ is a singular point, viz. p_0 contains x as a factor, and we have to consider the developments of y which belong to the domain of this singular point $x=0$.

It is convenient to change the form of the differential equation by dividing the whole equation by the first coefficient p_0, and then expanding each of the quotients $\frac{p_1}{p_0}, \frac{p_2}{p_0}, \ldots, \frac{p_m}{p_0}$ in a series of ascending powers of x. The new form is

$$\frac{d^m y}{dx^m} + p_1 \frac{d^{m-1} y}{dx^{m-1}} + \ldots + p_m y = 0,$$

or say

$$P(y) = (1, p_1, \ldots, p_m \between \frac{d}{dx}, 1)^m y = 0,$$

where $p_1, p_2, \ldots, p_m$ now denote each of them a series (finite or infinite) of integer powers of x, but containing only a finite number of negative powers of x.

5. Such an equation frequently admits of a "regular integral" of the form $y = x^\rho E(x)$, where $E(x)$ is a series of positive integer powers $E_0 + E_1 x + E_2 x^2 + \ldots$, ($E_0$ not $=0$, for this would imply a different value of ρ)*. To determine whether

* The expression regular integral is afterwards used in a more general sense, see *post*, No. 10.

this is so, we substitute in the differential equation for y the value in question $x^\rho E(x)$, thus obtaining a series

$$\Omega_0 x^{\rho-\theta} + \Omega_1 x^{\rho-\theta+1} + \ldots,$$

(where θ is a determinate positive integer depending on the negative powers of x in the equation); the coefficients Ω_0, $\Omega_1, \ldots$ are functions of ρ of an order not exceeding m, and contain also the coefficients E_0, E_1, $E_2, \ldots$ linearly; in particular, Ω_0 contains E_0 as a factor, say its value is $= E_0 \Pi_0$. The series should vanish identically. Supposing that Π_0 contains ρ, then we have $\Pi_0 = 0$, an equation of an order not exceeding m for the determination of ρ. For any root $\rho = \rho_0$ of this equation, E_0 remains arbitrary and may be taken $= 1$; the equations $\Omega_1 = 0$, $\Omega_2 = 0, \ldots$ then serve to determine the ratios to E_0 of the remaining coefficients E_1, $E_2, \ldots$; and we thus have the solution $y = x^{\rho_0}(1 + E_1 x + E_2 x^2 + \ldots)$, where ρ_0 and the coefficients have determinate values.

6. I stop to notice a curious form of illusory solution; the assumed form of solution is

$$y = x^\rho (\ldots + E_{-2}x^{-2} + E_{-1}x^{-1} + E_0 + E_1 x + \ldots),$$

the series being a double series extending both ways to infinity, or say a back-and-forward series; we have here a series of equations

$$\ldots \Omega_{-2} = 0, \quad \Omega_{-1} = 0, \quad \Omega_0 = 0, \quad \Omega_1 = 0, \ldots,$$

which leave ρ undetermined, but determine the ratios of the several coefficients to one of these coefficients, say E_0; or taking this $= 1$, we have a solution

$$y = x^\rho (\ldots + E_{-2}x^{-2} + E_{-1}x^{-1} + 1 + E_1 x + E_2 x^2 + \ldots)$$

where the coefficients are determinate functions of the arbitrary symbol ρ. Such a series is in general divergent for all values of the variable, and thus is altogether without meaning. As a simple instance, take the differential equation $\dfrac{dy}{dx} - y = 0$, which is satisfied by

$$y = \left\{\ldots (\rho - 1)\rho x^{\rho-2} + \rho x^{\rho-1} + x^\rho + \frac{x^{\rho+1}}{\rho+1} + \ldots\right\};$$

see my paper, Cayley, Note on Riemann's paper, "Versuch einer allgemeinen Auffassung der Integration und Differentiation," *Werke*, pp. 331—344; *Math. Ann.* t. XVI. (1880), pp. 81, 82), [751].

7. A more general form of integral is Thomé's "normal elementary integral," $y = e^w x^\rho E(x)$, where w is $=$ a finite series $\dfrac{c_{\alpha-1}}{x^{\alpha-1}} + \ldots + \dfrac{c_1}{x}$ of negative powers of x (α a positive integer, $= 2$ at least). To discover whether such a form exists, observe that, writing for a moment $\dfrac{dy}{dx} = y'$, and so for the other symbols, we have $\dfrac{y'}{y} = w' + \dfrac{\rho}{x} + \dfrac{E'(x)}{E(x)}$,

where the last term $\dfrac{E'(x)}{E(x)}$ may be expanded as a series of positive integer powers of x, so that, writing $\dfrac{y'}{y}=z$, we have

$$z=-\frac{(\alpha-1)\,c_{\alpha-1}}{x^\alpha}-\ldots-\frac{c_1}{x^2}+\frac{\rho}{x}+G_0+G_1x+\ldots;$$

we can, by introducing into the differential equation the new variable $z\left(=\dfrac{y'}{y}\right)$ in place of y, obtain for z a differential equation (not a linear one) of the order $m-1$, and this should be satisfied by the series in question, viz. a series containing a finite number of negative powers of x; we endeavour to determine the first term $\dfrac{-(\alpha-1)\,c_{\alpha-1}}{x^\alpha}$, or say $\dfrac{A}{x^\alpha}$, where the exponent α is a positive integer which is to be determined, and A is not $=0$; this is done by a well-known process; we write in the equation $z=\dfrac{A}{x^\alpha}$, and (if possible to do so) determine α (a positive integer $=2$ at least) by the condition that two or more of the terms shall have the same negative index $-p$ preceding in order all the other indices, viz. the absolute value of p must be greater than the absolute value of any other negative index; we then equate to zero the whole coefficient of the term in question x^{-p}, and thus obtain a value not $=0$ of A. And having thus obtained the first term $\dfrac{A}{x^\alpha}$, we can, by assuming the form

$$z=\frac{A}{x^\alpha}+\frac{B}{x^{\alpha-1}}+\ldots+\frac{K}{x^2}+\frac{L}{x}+\ldots$$

with indeterminate coefficients, and substituting in the equation, find the remaining coefficients $B,\ldots, K, L$; we then have

$$c_{\alpha-1}=\frac{-A}{\alpha-1},\ \ldots,\ c_1=-K,$$

giving the value $\dfrac{c_{\alpha-1}}{x^{\alpha-1}}+\ldots+\dfrac{c_1}{x}$ of w; instead of determining the subsequent coefficients from the z-equation, we may, from the original equation, writing therein $y=e^wY$, obtain an equation for $Y(=e^{-w}y)$ which will be linear of the order m, and will admit of a regular integral $Y=x^\rho E(x)$. Or we may, going on a step further with the z-equation, find therefrom the coefficient L which is $=\rho$.

8. As an example, take the equation

$$\frac{d^2y}{dx^2}+\left(\frac{3}{x}-\frac{1}{x^2}\right)\frac{dy}{dx}+\frac{1}{x^2}y=0,$$

which has the elementary normal integral

$$y=e^{-\frac{1}{x}}\frac{1}{x}.$$

To investigate this, writing $\frac{y'}{y}=z$, we have $\frac{y''}{y}=z'+z^2$, and thence the z-equation

$$\frac{dz}{dx}+z^2+\left(\frac{3}{x}-\frac{1}{x^2}\right)z+\frac{1}{x^2}=0.$$

Substituting herein the value $z=\frac{A}{x^\alpha}$, we obtain the function

$$-\frac{A\alpha}{x^{\alpha+1}}+\frac{A^2}{x^{2\alpha}}-\frac{A}{x^{\alpha+2}}+\frac{1}{x^2},$$

and here the value $\alpha=2$ gives the two terms $\frac{A^2}{x^4}$ and $\frac{-A}{x^4}$, each with a negative index -4, the absolute value whereof is greater than the absolute values of the other indices; we then have $A^2-A=0$, giving a value $A=1$, which is not $=0$; and we have thus a first term $z=\frac{1}{x^2}$; the complete value is, as it happens, the finite series $z=\frac{1}{x^2}-\frac{1}{x}$; and z being $=\frac{y'}{y}$, we thence obtain the integral in question, $y=e^{-\frac{1}{x}}\frac{1}{x}$.

9. Returning to the question of the regular integrals, we may consider the function

$$P(x^\rho),\ =(1,\ p_1,\ldots,\ p_m \between \frac{d}{dx},\ 1)^m\, x^\rho,$$

say this is the so-called "determinirende Function," but I will call it the Indicial function; this is a function of (x, ρ); we have therein terms arising from

$$\left(\frac{d}{dx}\right)^m x^\rho,\quad \left(\frac{d}{dx}\right)^{m-1} x^\rho,\ldots,$$

viz. these are equal to $[\rho]^m x^{\rho-m}$, $[\rho]^{m-1}x^{\rho-m-1},\ldots$ respectively, but these will be multiplied by powers of x contained in the coefficients $p_1, p_2, \ldots, p_m$ respectively; the function is thus of the degree m as regards ρ, but the coefficient of any power of x is of the degree m, or of some inferior degree, according to the terms $[\rho]^m$, $[\rho]^{m-1},\ldots$ which enter into the coefficient. The coefficient of the lowest power of x in the indicial function may be termed the indicial coefficient, or simply the indicial; and the equation obtained by equating this coefficient to zero the indicial equation. By what precedes, the indicial is a function of ρ, which is of the degree m at most, but which may be of any inferior degree, or even be an absolute constant. Hence, considering a regular integral $x^\rho E(x)$, the index ρ is determined by the indicial equation, and to each root of this equation there corresponds in general a regular integral $x^\rho E(x)$; the number of regular integrals is thus at most $=m$.

10. Suppose, first, that the indicial equation is of the degree m, and that the roots of this equation are all unequal; there are, in this case, m values, say $\rho_1, \rho_2, \ldots, \rho_m$ of ρ, and each of these gives rise to a regular integral $x^\rho E(x)$, viz. there will be corresponding to each value of ρ a series of positive integer powers $E(x)$, which will be

a convergent series for sufficiently small values of x. Some of these series may be finite series, viz. this will be the case if the indicial equation has roots differing from each other by integer values (but a finite series may of course be regarded as convergent); and in the case of two or more equal roots, it is necessary to extend the notion of a regular integral by including under it series multiplied into positive integer powers of $\log x$; but, with this modification, the conclusion holds good; if the indicial equation be of the degree m, the number of regular integrals will always be $=m$, viz. there will be m integrals involving series $E(x)$, or, it may be, $E(x)(\log x)^k$, where each series $E(x)$ is a convergent series for sufficiently small values of x.

11. But suppose the indicial equation is of an order $m-\gamma$ inferior to m, and to fix the ideas suppose that the roots are all unequal. There are in this case $m-\gamma$ values, say $\rho_1, \rho_2, \ldots, \rho_{m-\gamma}$ of ρ, and each of these will apparently give rise to a regular integral $x^\rho E(x)$; but it may be that in any such case the series $E(x)$ is a series divergent for all values, however small, of the variable x, and that we thus have in appearance only, but not in reality, a regular integral; or say the integral is illusory. The conclusion is that the indicial equation being of a degree $m-\gamma$ inferior to m, the number of regular integrals is at most $=m-\gamma$; but it may have any less value than $m-\gamma$, or even there may be no regular integral. It is hardly necessary to remark that, if the indicial be an absolute constant, or say if the degree of the indicial equation be $=0$, then there is no value of ρ, and consequently no regular integral.

12. By way of illustration, consider first the differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{x}y=0,$$

for which the degree of the indicial equation is equal to the order of the equation $(=2)$; viz. the indicial function is

$$=\rho(\rho-1)x^{\rho-2}+\rho x^{\rho-2}+x^{\rho-1},$$

and the indicial equation is thus $\rho(\rho-1)+\rho=0$, that is, $\rho^2=0$, viz. there are here two roots, each $=0$.

It is convenient not in the first instance to write $\rho=0$, but to substitute in the equation the value

$$y=Ax^\rho+Bx^{\rho+1}+Cx^{\rho+2}+\ldots.$$

We thus find

$$\begin{array}{rccc} & x^{\rho-2}, & x^{\rho-1}, & x^{\rho},\ldots \\ \hline 0= & \rho\,.\,\rho-1\,.\,A, & \rho+1\,.\,\rho\,.\,B, & \rho+2\,.\,\rho+1\,.\,C \\ & +\quad \rho A, & +\rho+1\,.\,B, & +\rho+2\,.\,C \\ & & +A, & +B, \end{array}$$

viz. we have the equations

$$\begin{aligned} \rho^2 A \qquad &=0,\\ (\rho+1)^2B+A&=0,\\ (\rho+2)^2C+B&=0, \end{aligned}$$

where observe that in each equation the function of ρ, which multiplies the posterior coefficient, is of a degree superior to that which multiplies the other coefficient. The first equation gives $\rho = 0$; A arbitrary, but say $A = 1$; the values of $B, C, \ldots$ are then

$$-\frac{1}{(\rho+1)^2}, \quad +\frac{1}{(\rho+1)^2(\rho+2)^2}, \quad -\frac{1}{(\rho+1)^2(\rho+2)^2(\rho+3)^2}, \ldots$$

giving rise to a series which in this form, and when ρ is put equal to its value, $\rho = 0$, is at once seen to be convergent (even for indefinitely large values of x, but this is an accident; it would have been enough if the series had been convergent for sufficiently small values), viz. the series is

$$y = 1 - \frac{x}{1^2} + \frac{x^2}{1^2 \,.\, 2^2} - \frac{x^3}{1^2 \,.\, 2^2 \,.\, 3^2} + \ldots.$$

There is another integral

$$y = \left(1 - \frac{x}{1^2} + \frac{x^2}{1^2 \,.\, 2^2} - \ldots\right) \log x + 2x - \tfrac{3}{4}x^2 + \tfrac{11}{108}x^3 - \tfrac{25}{3456}x^4 + \ldots,$$

involving $\log x$ and a second series which is also convergent.

13. But consider next the before-mentioned equation

$$\frac{d^2y}{dx^2} + \left(\frac{3}{x} - \frac{1}{x^2}\right)\frac{dy}{dx} + \frac{1}{x^2}y = 0,$$

for which the degree of the indicial equation is equal to order $-1 (=1)$; in fact, here the indicial function is

$$= \rho(\rho-1)x^{\rho-2} + 3\rho x^{\rho-2} - \rho x^{\rho-3} + x^{\rho-2},$$

and the indicial is thus $= -\rho$, giving the index $\rho = 0$. There is thus at most one regular integral $E(x)$, but this is in fact an illusory one. Substituting in the equation the value

$$y = Ax^\rho + Bx^{\rho+1} + Cx^{\rho+2} + \ldots,$$

we find

	$x^{\rho-3}$,	$x^{\rho-2}$,	$x^{\rho-1}$,	x^{ρ}.
$0 =$		$\rho(\rho-1)A$,	$(\rho+1)\rho B$,	$(\rho+2)(\rho+1)C$,
	$-\rho A$,	$-(\rho+1)B$,	$-(\rho+2)C$,	$-(\rho+3)D$,
		$+3\rho A$,	$+3(\rho+1)B$,	$+3(\rho+2)C$,
		$+A$,	$+B$,	$+C$,

and we thus have the equations

$$\begin{aligned} \rho A \qquad\qquad &= 0, \\ (\rho+1)^2 A - (\rho+1) B &= 0, \\ (\rho+2)^2 B - (\rho+2) C &= 0, \\ (\rho+3)^2 C - (\rho+3) D &= 0, \end{aligned}$$

where observe that in each equation the function of ρ, which multiplies the posterior coefficient, is of a degree inferior to that which multiplies the other coefficient. The first equation then gives $\rho=0$; A arbitrary, or say $=1$: the other coefficients B, C,... are then found to be

$$\rho+1,\quad (\rho+1)(\rho+2),\quad (\rho+1)(\rho+2)(\rho+3),\ldots$$

or substituting for ρ its value $=0$, the series is

$$1+1x+1\,.\,2\,.\,x^2+1\,.\,2\,.\,3x^3+\ldots,$$

which is divergent for any value whatever, however small, of x; or the integral is as mentioned an illusory one.

14. The last-mentioned differential equation is *reducible*, viz. we have

$$P(y)=\left\{\frac{d^2}{dx^2}+\left(\frac{3}{x}-\frac{1}{x^2}\right)\frac{d}{dx}+\frac{1}{x^2}\right\}y=\left(\frac{d}{dx}+\frac{2}{x}\right)\left(\frac{d}{dx}+\frac{1}{x}-\frac{1}{x^2}\right)y, \text{ say } =QDy,$$

where of course the order of the factors Q, D is material.

It is a general theorem that the indicial, belonging to the product P, is of a degree equal to the sum of the degrees of the indicials, belonging to the factors Q, D respectively; and, in fact, the indicial of P is (as was seen) of the degree 1; and the indicials of Q, D are of the degrees 1, 0 respectively. But, moreover, if the indicial of P is of a degree $m-\gamma$, less than m the order of the equation (in the present case $m=2$, $m-\gamma=1$), then, in order that the equation may have $m-\gamma$ regular integrals, it is a necessary and sufficient condition that P shall be decomposable into a product QD, where the factors Q, D (being functions of the same form as P) are of the orders γ and $m-\gamma$ respectively, and where, moreover, the *second* factor D shall have an indicial of the degree $m-\gamma$, and consequently the *first* factor Q an indicial of the degree zero. In the present case, it is the second factor which has an indicial of degree zero; and thus the condition is not satisfied. And accordingly the equation ought not to have $(m-\gamma=)$ 1 regular integral; and we have seen that there is in fact no regular integral. The investigation serves to show in what sense it is that there is no regular integral, or generally in what sense it is that the number of regular integrals may be less than $m-\gamma$.

15. The theorem may, it appears to me, be stated in the more general form: the necessary and sufficient condition that the differential equation $P(y)=0$ of the order m, but having an indicial of the degree $m-\gamma$, shall have $m-\gamma-\delta$ regular integrals is that the function P shall be a product of the form $P=QMD$, where the orders of Q, M, D are $\gamma+\delta-\theta$, θ, $m-\gamma-\delta$ respectively, and the degrees of their indicials δ, 0, $m-\gamma-\delta$ respectively; or, to denote this in a compendious manner, say

$$\overset{m}{\underset{m-\gamma}{P}}=\overset{\gamma+\delta-\theta}{\underset{\delta}{Q}}\;\overset{\theta}{\underset{0}{M}}\;\overset{m-\gamma-\delta}{\underset{m-\gamma-\delta}{D}};$$

θ is an integer which is not $=0$, and which is at most $=\gamma$, for otherwise the function

Q of the order $\gamma+\delta-\theta$ could not have an indicial of the degree δ. If there is no regular integral, then $m=\gamma+\delta$, D is a mere constant or say unity, and the formula is

$$\overset{m}{\underset{m-\gamma}{P}} = \overset{m-\theta}{\underset{m-\gamma}{Q}}\ \overset{\theta}{\underset{0}{M}},$$

where θ is not $=0$, and is at most $=m-\gamma$.

16. For differential equations of the form $P(y)=0$ above considered, Floquet gave a theorem, which as afterwards remarked by him is not generally true, viz. this was $P=ABC\ldots$, a product of linear factors of the form $\frac{d}{dx}+\Sigma_{-\infty}^{\infty}C_i x^i$ (i an integer). The question of decomposition is considered in my next following paper "On Linear Differential Equations (the Theory of Decomposition)," [851], and by means of the formulæ there given (and by the process indicated *ante* No. 7), it could be decided whether any particular differential equation admits of a decomposition $P=ABC\ldots$, where the linear factors are functions *not* of the form just referred to, but of the form $\frac{d}{dx}+\Sigma C_i x^i$, in which the series contains only a finite number of negative powers, say this is a decomposition into linear regular factors.