848.

ON THE TRANSFORMATION OF THE DOUBLE-THETA FUNCTIONS.

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I PROPOSE to reproduce Hermite's Memoir "Sur la théorie de la transformation des fonctions Abéliennes," *Comptes Rendus*, t. XL. (1855), pp. 249, ..., 784, with some changes of notation and developments. Hermite's functions are even or odd according as we have $\mu q + \nu p$ even or odd; viz. his characteristic is $\begin{pmatrix}\mu, & \nu\\ q, & p\end{pmatrix}$, or the letters p, q, are misplaced; I write, therefore, r instead of p, so as to have the characteristic $\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}$; and then for symmetry it is necessary to interchange the suffixes 2, 3 and the letters c, d; the invariant function of the periods, instead of being as with him

$$\omega_0 v_3 - \omega_3 v_0 + \omega_1 v_2 - \omega_2 v_1,$$

must be taken to be

$$\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1.$$

Moreover, I write A, B for his G, G', so as, instead of

$$(G, H, G' \between x, y)^2,$$

to have in the expressions of the theta-functions the quadric function $(A, H, B \between x, y)^2$; and I alter the arrangement of the memoir so as to separate more completely the preliminary theory from the theory of the transformation.

GENERAL THEORY. Art. Nos. 1 to 21 (*several sub-headings*).

The functions Π {K, indef. or def.}.

1. Consider a function

$$\Pi \begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix} (x, y)(A, H, B)\{K, \text{indef. or def.}\},$$

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having: the characteristic $\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}$, where the characters μ, ν, q, r are positive or negative integers, which may be taken to have each of them only the values (0, 1) at pleasure, so that the number of functions is 2^4, $=16$;

the arguments (x, y);

the parameters, or conjoint quarter-periods, (A, H, B);

and the potency K, a positive integer;

and which is either indefinite or definite, as will be explained; the function, moreover, contains linearly certain arbitrary constants, the number of them depending on the value of K, as will be explained.

The function may be written $\Pi(x, y)$ or in any other less abbreviated form which may be convenient.

2. The function $\Pi(x, y)\,\{K \text{ indef.}\}$ is defined by the following four equations:

$$\begin{aligned}
\Pi(x+1, y) &= (-)^{\mu}\,\Pi(x, y),\\
\Pi(x, y+1) &= (-)^{\nu}\,\Pi(x, y),\\
\Pi(x+A, y+H) &= (-)^{q}\,\Pi(x, y)\,\text{exp.}-2\pi K(2x+A),\\
\Pi(x+H, y+B) &= (-)^{r}\,\Pi(x, y)\,\text{exp.}-2\pi K(2y+B),
\end{aligned}$$

and the function $\Pi(x, y)\,\{K \text{ def.}\}$ by the same equations, together with the following fifth equation,

$$\Pi(-x, -y) = (-)^{\mu q+\nu r}\,\Pi(x, y);$$

viz. the definite function is an even function or else an odd function of the arguments according as $\mu q+\nu r$ is even or odd. We may call $\mu q+\nu r$ the index; and the function is then even or odd according as the index is even or odd.

It is perhaps worth noticing that it would be allowable to define a function $\Pi(x, y)\,\{K, \text{ skew def.}\}$, by the corresponding relation

$$\Pi(-x, -y) = -(-)^{\mu q+\nu r}\,\Pi(x, y),$$

but I do not propose here to develope this notion.

3. The four equations give rise to the following one,

$$\begin{aligned}
&\Pi(x+a_0+Aa_2+Ha_3,\ y+a_1+Ha_2+Ba_3)\\
&\quad= (-)^{\mu a_0+\nu a_1+q a_2+r a_3}\,\Pi(x, y)\times\text{exp.}-2\pi K\,\{2a_2x+2a_3y+(A, H, B\between a_2, a_3)^2\},
\end{aligned}$$

where a_0, a_1, a_2, a_3 are any positive or negative integers (zero not excluded), and which single equation, in fact, includes the preceding four equations.

4. In regard to the parameters it is to be observed that, if $A, H, B = A_0+i\alpha$, $H_0+i\eta$, $B_0+i\beta$, we must have (α, η, β) a determinate positive quadratic form; viz. this is the necessary and sufficient condition for the convergence of the series for the development of the function.

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5. The number of arbitrary constants is for the indefinite function $=K^2$; but for the definite function it is, when K is odd, $=\frac{1}{2}(K^2+1)$; and when K is even, it is $=\frac{1}{2}K^2$, except in the case of a characteristic $\begin{pmatrix} 0, & 0 \\ q, & r \end{pmatrix}$, when it is $=\frac{1}{2}(K^2+4)$.

In particular, for $K=1$, there is only a single arbitrary constant, which is a mere factor of the function; taking it to be $=1$, as presently explained, we have the 16 theta-functions.

6. The function $\Pi(x, y)$ is developed in a series of exponentials, in the form

$$\Pi(x, y) = \Sigma\, (-)^{mq+nr} A_{m,\,n} \text{ exp. } i\pi \left\{(2m+\mu)\,x + (2n+\nu)\,y + \frac{1}{4K}(A,\ H,\ B \between 2m+\mu,\ 2n+\nu)^2\right\},$$

where m and n have each of them all positive and negative integer values (zero not excluded) from $-\infty$ to ∞. In fact, substituting this series in the four equations, they are all of them satisfied if only

$$A_{m+K,\,n} = A_{m,\,n}\ ;\quad A_{m,\,n+K} = A_{m,\,n}\,.$$

Consequently the following K^2 coefficients remain arbitrary, viz. those with the suffixes

$$\begin{array}{c|ccc} & 0, & 1, \ldots & K-1\,; \\ \hline 0 & " & " & " \\ 1 & & & \\ \vdots & & & \\ K-1 & " & " & " \end{array}$$

and we have for $\Pi(x, y)$ a sum of K^2 terms, each a determinate series multiplied into one of the arbitrary coefficients $A_{0,\,0}$, $A_{0,\,1}$, &c. The indefinite function thus contains, as already mentioned, K^2 arbitrary constants.

7. Substituting in the fifth equation, we have for the definite function the further condition

$$A_{-m-\mu,\,-n-\nu} = A_{m,\,n},$$

which it is clear will be satisfied generally if only it is satisfied by the coefficients in the foregoing set of K^2 coefficients.

8. In the case K odd, we thus reduce the number of arbitrary coefficients to $\frac{1}{2}(K^2+1)$; the mode in which this takes place is best seen by an example. Suppose $K=3$, so that $A_{m+3,\,n} = A_{m,\,n}$; $A_{m,\,n+3} = A_{m,\,n}$. For the coefficients of the indefinite function, the suffixes are

00, 01, 02,

10, 11, 12,

20, 21, 22.

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And if we suppose $\mu=0$, $\nu=1$, then the new condition is $A_{-m,\,-n-1}=A_{m,\,n}$, viz. writing down only the suffixes, we thus obtain

0, 0 = 0, − 1	1, 0 = − 1, − 1	2, 0 = − 2, − 1,
0, 1 = 0, − 2	1, 1 = − 1, − 2	2, 1 = − 2, − 2,
0, 2 = 0, − 3	1, 2 = − 1, − 3	2, 2 = − 2, − 3,

that is,

0, 0 = 0, 2	1, 0 = 2, 2	2, 0 = 1, 2,
0, 1 = 0, 1	1, 1 = 2, 1	2, 1 = 1, 1,
0, 2 = 0, 0	1, 2 = 2, 0	2, 2 = 1, 0,

viz. one of these equations 0, 1 = 0, 1 is an identity, but the other equations occur each twice; or we have four equations, each of them an equality between two out of the remaining 8 coefficients; the number of arbitrary coefficients is thus $1+\tfrac{1}{2}(9-1), =5$; and so in general the number is

$$1+\tfrac{1}{2}(K^2-1), \quad =\tfrac{1}{2}(K^2+1).$$

9. When K is even, it is necessary to distinguish between the case $(\mu, \nu)=(0, 0)$ and the remaining three cases $(\mu, \nu)=(1, 0)$, $(0, 1)$ or $(1, 1)$. In the former case, the relation between the coefficients is $A_{-m,\,-n}=A_{m,\,n}$; there are four identities, $0, 0=0, 0$; $0, \tfrac{1}{2}K=0, \tfrac{1}{2}K$; $\tfrac{1}{2}K, 0=\tfrac{1}{2}K, 0$; $\tfrac{1}{2}K, \tfrac{1}{2}K=\tfrac{1}{2}K, \tfrac{1}{2}K$; and the remaining K^2-4 equations occur each twice, that is, we have $\tfrac{1}{2}(K^2-4)$ equations, each of them an equality between two of the remaining K^2-4 coefficients; the number of arbitrary coefficients is thus $4+\tfrac{1}{2}(K^2-4), =\tfrac{1}{2}(K^2+4)$.

In the latter case, there are no identities and the K^2 equations occur each twice, that is, we have $\tfrac{1}{2}K^2$ equations, each of them an equality between two of the K^2 coefficients; and we thus have $\tfrac{1}{2}K^2$ arbitrary coefficients.

10. Recapitulating, it thus appears that

for an indefinite function, the number of coefficients $=K^2$;

for a definite function, the number $=\tfrac{1}{2}(K^2+1)$, K odd;

$=\tfrac{1}{2}K^2$, K even, and $(\mu, \nu)=(1, 0)$, $(0, 1)$ or $(1, 1)$;

$=\tfrac{1}{2}(K^2+4)$, K even, and $(\mu, \nu)=(0, 0)$.

The Theta-functions.

11. In the particular case $K=1$, the distinction between the indefinite function and the definite function disappears, and we have instead of $\Pi(x, y)$, the theta-functions $\Theta(x, y)$, satisfying the four equations

$$\begin{aligned}
\Theta(x+1, y) &= (-)^{\mu}\,\Theta(x, y),\\
\Theta(x, y+1) &= (-)^{\nu}\,\Theta(x, y),\\
\Theta(x+A, y+H) &= (-)^{q}\,\Theta(x, y)\exp. -i\pi(2x+A),\\
\Theta(x+H, y+B) &= (-)^{r}\,\Theta(x, y)\exp. -i\pi(2y+B),
\end{aligned}$$

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and the fifth equation

$$\Theta(-x, -y) \qquad = (-)^{\mu q + \nu r}\,\Theta(x, y);$$

as before, $\mu q + \nu r$ is the index of the function.

The four equations are all of them included in the following one:

$$\Theta(x + a_0 + Aa_2 + Ha_3,\ y + a_1 + Ha_2 + Ba_3) = (-)^{\mu a_0 + \nu a_1 + q a_2 + r a_3}\,\Theta(x, y)$$
$$\times \exp. - i\pi\,\{2a_2 x + 2a_3 y + (A,\ H,\ B \between a_2,\ a_3)^2\},$$

where a_0, a_1, a_2, a_3 are each of them any positive or negative integer, zero not excluded.

Moreover, we take $A_{0,0} = 1$, and the value of the function thus is

$$\Theta(x, y) = \Sigma\,(-)^{mq+nr} \exp. i\pi\,\{(2m+\mu)\,x + (2n+\nu)\,y + \tfrac{1}{4}\,(A,\ H,\ B \between 2m+\mu,\ 2n+\nu)^2\}.$$

12. The sum of two characteristics is the characteristic obtained by taking the sums of the component terms or characters,

$$\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix} + \begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix} = \begin{pmatrix}\mu+\mu', & \nu+\nu'\\ q+q', & r+r'\end{pmatrix};$$

and similarly for any number of characteristics. I use the sign $=$, but this properly denotes a congruence, mod. 2; and the like as regards the indices.

The sum of two identical characteristics, or generally of any number of characteristics taken each of them any even number of times, is $= \begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$. And this characteristic $\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ may be called the characteristic 0.

It should be observed, that the index of the sum is not in general equal to the sum of the indices. To make it so, we must have, for two characteristics,

$$(\mu+\mu')(q+q') + (\nu+\nu')(r+r') = \mu q + \nu r + \mu' q' + \nu' r',$$

that is,

$$\mu q' + \mu' q + \nu r' + \nu' r = 0;$$

and there is obviously a like formula for the case of more than two characteristics.

Two or more characteristics, such that they have the sum of the indices equal to the index of the sum, are said to be "in direct relation" or "directly related" to each other. The sum of the indices and the index of the sum may differ by unity and we then have the inverse relation; but I do not propose to consider this.

13. Consider any number K of theta-functions, of the same arguments and parameters, but with the same or different characteristics. The product of these functions is in general a function $\Pi(x, y)\,\{K \text{ indef.}\}$, having a characteristic which is $=$ the sum of the characteristics of the theta-functions. In fact, from the four

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equations of the theta-functions, we at once obtain for their products $\Pi(x, y)$ the four equations

$$\begin{aligned}\Pi(x+1, y) &= (-)^{\Sigma\mu}\Pi(x, y),\\ \Pi(x, y+1) &= (-)^{\Sigma\nu}\Pi(x, y),\\ \Pi(x+A, y+H) &= (-)^{\Sigma q}\Pi(x, y)\exp. -i\pi K(2x+A),\\ \Pi(x+H, y+B) &= (-)^{\Sigma r}\Pi(x, y)\exp. -i\pi K(2y+B),\end{aligned}$$

which proves the theorem.

14. But if the indices are in direct relation to each other, then we have further

$$\Pi(-x, -y) = (-)^{\Sigma\mu\Sigma q+\Sigma\nu\Sigma r}\Pi(x, y),$$

and the product is thus a function $\Pi(x, y)$ {K, def.}.

15. Take the square of a theta-function, the characteristic is $=\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ or 0, and we have also twice the index $=0$; viz. the theta-function is in direct relation with itself. Hence the squared function is a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}(x, y)$ {2, def.}, and as such it contains linearly $\frac{1}{2}(2^2+4)$, $=4$ arbitrary constants. Hence, taking any five squares, since each of them is a function of the form in question, it follows that the squares of the 5 theta-functions are connected by a linear relation.

Göpel's relation between 4 theta-functions.

16. We may in a variety of ways (in fact, in 60 ways, as will presently be shown) select four theta-functions, all of them even, or else two of them even and two odd (that is, having the sum of their indices $=0$), such that the sum of their characteristics is $=0$; for instance, the functions may be

$$P'=\begin{pmatrix}0, & 0\\ 1, & 1\end{pmatrix},\quad P''=\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix},\quad S'=\begin{pmatrix}1, & 1\\ 0, & 1\end{pmatrix},\quad S''=\begin{pmatrix}1, & 1\\ 1, & 0\end{pmatrix},$$

indices

$$0\ ,\qquad 0\ ,\qquad 1\ ,\qquad 1\ .$$

The functions are thus in direct relation, and the product of the four functions is a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ {4, def.}. But obviously any one of the functions taken four times, or any two of them taken each twice, are in like manner four functions in direct relation, or the fourth powers P'^4, P''^4, S'^4, S''^4, and the squared products $P'^2P''^2$, $S'^2S''^2$, $P'^2S'^2$, $P'^2S''^2$, $P''^2S'^2$, $P''^2S''^2$, are in like manner each of them a function $\Pi\begin{pmatrix}0, & 0\\ 0, & 0\end{pmatrix}$ {4, def.}, viz. we have thus in all $1+4+6$, $=11$ such functions. But the Π function contains only $\frac{1}{2}(4^2+4)$, $=10$ arbitrary constants; hence there must be a linear relation between the 11 powers and products, and this is Göpel's relation.

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17. Starting with any two characteristics a, b at pleasure, the remaining characteristics form seven pairs, such that

$$a+b=c+d=e+f=g+h=i+j=k+l=m+n=o+p;$$

but among the seven pairs we have only three, suppose $c+d$, $e+f$, $g+h$, which are such that (a, b, c, d), (a, b, e, f), (a, b, g, h) are each of them either all even or else two even and two odd; that is, starting with any pair (a, b), we have these three tetrads having each of them the required property. The number of pairs (a, b) is $\frac{1}{2}16.15, =120$; and we thence derive $120\times 3, =360$ tetrads; but each such tetrad is of course derivable from any one of the six pairs contained in it; or the number of distinct tetrads is $\frac{1}{6}360, =60$, viz. we have, as mentioned above, 60 Göpel-tetrads.

The four functions Π_0, Π_1, Π_2, Π_3.

18. We consider four theta-functions θ_0, θ_1, θ_2, θ_3, which are such that to the modulus 2, the sum of the characters is $\equiv 0$, and also the sum of the indices is $\equiv 0$; taking the characters to be

$$\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix},\quad \begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix},\quad \begin{pmatrix}\mu'', & \nu''\\ q'', & r''\end{pmatrix},\quad \begin{pmatrix}\mu''', & \nu'''\\ q''', & r'''\end{pmatrix},$$

and writing throughout = for $\equiv$ (mod. 2), we have

$$\mu+\mu'+\mu''+\mu'''=0,$$
$$\nu+\nu'+\nu''+\nu'''=0,$$
$$q+q'+q''+q'''=0,$$
$$r+r'+r''+r'''=0,$$
$$\mu q+\nu r+\mu' q'+\nu' r'+\mu'' q''+\nu'' r''+\mu''' q'''+\nu''' r'''=0.$$

Writing for shortness $(01)=\mu q'+\mu' q+\nu r'+\nu' r$, and so in other cases; and further $(01)+(02)+(12)=(012)$, &c., then substituting for μ''', ν''', q''', r''' their values from the first four equations, we deduce $(012)=0$; and similarly $(013)=0$, $(023)=0$, $(123)=0$.

19. Consider now a product $\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}}$, where $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}$ is = a given odd number k; the characteristic is

$$\begin{pmatrix}\mu\mathrm{a}+\mu'\mathrm{b}+\mu''\mathrm{c}+\mu'''\mathrm{d}, & \nu\mathrm{a}+\nu'\mathrm{b}+\nu''\mathrm{c}+\nu'''\mathrm{d}\\ q\mathrm{a}+q'\mathrm{b}+q''\mathrm{c}+q'''\mathrm{d}, & r\mathrm{a}+r'\mathrm{b}+r''\mathrm{c}+r'''\mathrm{d}\end{pmatrix},$$

and it hence follows that the index is

$$=\mathrm{a}(\mu q+\nu r)+\mathrm{b}(\mu' q'+\nu' r')+\mathrm{c}(\mu'' q''+\nu'' r'')+\mathrm{d}(\mu''' q'''+\nu''' r''').$$

In fact, forming the index in question, we have first terms in a^2, b^2, c^2, d^2, which upon writing therein a, b, c, d for these values respectively ($\mathrm{a}^2=\mathrm{a}$, &c.) give the

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required value; we have therefore only to show that the sum of the remaining terms in ab, &c., is $=0$. These terms are

$$\text{ab}\,(01)+\text{ac}\,(02)+\text{ad}\,(03)+\text{bc}\,(12)+\text{bd}\,(13)+\text{cd}\,(23),$$

and writing herein $\text{a}+\text{b}+\text{c}+\text{d}=1$, and thence $\text{ad}=\text{a}(1-\text{a}-\text{b}-\text{c})$, $=\text{ab}+\text{ac}$, and similarly $\text{bd}=\text{ab}+\text{bc}$, $\text{cd}=\text{ac}+\text{bc}$, the terms in question become

$$=\text{ab}\,(013)+\text{ac}\,(023)+\text{bc}\,(123),$$

that is, they become $=0$.

20. We thus see that the function $\theta_0{}^{\text{a}}\theta_1{}^{\text{b}}\theta_2{}^{\text{c}}\theta_3{}^{\text{d}}$ has an index which is

$$=\text{a ind }\theta_0+\text{b ind }\theta_1+\text{c ind }\theta_2+\text{d ind }\theta_3.$$

Consider separately four products $\theta_0{}^{\text{a}}\theta_1{}^{\text{b}}\theta_2{}^{\text{c}}\theta_3{}^{\text{d}}$, in which the exponents a, b, c, d satisfy successively the relations (always to modulus 2)

$$\begin{array}{ll} \text{b}+\text{d}=0, & \text{c}+\text{d}=0, \\ \text{b}+\text{d}=1, & \text{c}+\text{d}=0, \\ \text{b}+\text{d}=0, & \text{c}+\text{d}=1, \\ \text{b}+\text{d}=1, & \text{c}+\text{d}=1. \end{array}$$

Combining herewith the relation $\text{a}+\text{b}+\text{c}+\text{d}=1$, it follows that the exponents a, b, c, d are

$$\begin{array}{llll} =\text{d}+1, & \text{d}\;, & \text{d}\;, & \text{d}\;, \\ \text{d}\;, & \text{d}+1, & \text{d}\;, & \text{d}\;, \\ \text{d}\;, & \text{d}\;, & \text{d}+1, & \text{d}\;; \\ \text{d}\;, & \text{d}\;, & \text{d}\;, & \text{d}+1, \end{array}$$

in the four cases respectively. Then substituting these values, the characteristics become

$$\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}, \quad \begin{vmatrix}\mu', & \nu' \\ q', & r'\end{vmatrix}, \quad \begin{vmatrix}\mu'', & \nu'' \\ q'', & r''\end{vmatrix}, \quad \begin{vmatrix}\mu''', & \nu''' \\ q''', & r'''\end{vmatrix},$$

viz. the four products have the same characters as θ_0, θ_1, θ_2, θ_3 respectively; and in like manner recollecting that

$$\text{ind }\theta_0+\text{ind }\theta_1+\text{ind }\theta_2+\text{ind }\theta_3=0,$$

we see that the four products have the same indices as θ_0, θ_1, θ_2, θ_3 respectively.

More generally write Π_0, Π_1, Π_2, $\Pi_3=\Sigma\theta_0{}^{\text{a}}\theta_1{}^{\text{b}}\theta_2{}^{\text{c}}\theta_3{}^{\text{d}}$, where for the four cases respectively the exponents a, b, c, d satisfy the conditions already referred to; then Π_0, Π_1, Π_2, Π_3 have the same characteristics, and the same indices, as θ_0, θ_1, θ_2, θ_3 respectively.

21. It can be shown that each of the functions Π contains $\frac{1}{2}(k^2+1)$ constants. It will be recollected that we have between θ_0, θ_1, θ_2, θ_3 an equation of the form

$$0=(\theta_0{}^4,\ \theta_1{}^4,\ \theta_2{}^4,\ \theta_3{}^4,\ \theta_0{}^2\theta_1{}^2,\ \theta_0{}^2\theta_2{}^2,\ \theta_0{}^2\theta_3{}^2,\ \theta_1{}^2\theta_2{}^2,\ \theta_1{}^2\theta_3{}^2,\ \theta_2{}^2\theta_3{}^2,\ \theta_0\theta_1\theta_2\theta_3);$$

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this serves to express, say θ_3^4, in lower powers of θ_3; and by successive applications of this equation, we can reduce $\Sigma\theta_0^{\rm a}\theta_1^{\rm b}\theta_2^{\rm c}\theta_3^{\rm d}$ to a form in which d has only one of the values 0, 1, 2 or 3; we do not by this transformation alter the suffix of Π, viz. a term originally of the form Π_0, Π_1, Π_2 or Π_3, will by the transformation give rise only to terms which are of the same form Π_0, Π_1, Π_2, Π_3 (as the case may be). The number of constants in Π_0 is thus

$$= \text{number of partitions of } k \text{ into four parts a, b, c, d,}$$

under the conditions

$$\text{d} = 0 \text{ or } 2;\ \text{a odd, b, c each even,}$$
$$\text{d} = 1 \text{ or } 3;\ \text{a even, b, c each odd,}$$

where, in reckoning the partitions, the order of the parts is taken into account: the partitions are thus as follows

$$\begin{aligned}
\text{d} = 0,&\quad (\text{a}-1) + \text{b} + \text{c} = k-1,\\
\text{d} = 1,&\quad \text{a} + (\text{b}-1) + (\text{c}-1) = k-3,\\
\text{d} = 2,&\quad (\text{a}-1) + \text{b} + \text{c} = k-3,\\
\text{d} = 3,&\quad \text{a} + (\text{b}-1) + (\text{c}-1) = k-5,
\end{aligned}$$

where the parts a or (a − 1), b or (b − 1), c or (c − 1), as the case may be, are all of them even; hence, writing $k' = \frac{1}{2}(k-1)$, the cases are

$$\text{a}' + \text{b}' + \text{c}' = k',\ k'-1,\ k'-1, \text{ or } k'-2,$$

where the a′, b′, c′ are odd or even (zero not excluded) at pleasure; as already mentioned, the order of the parts is taken into account: thus the particulars of 3 would be

300,	210,	120,	030,	No. is 10, $=\frac{1}{2}4 \cdot 5$.
	201,	111,	021,	
		102,	012,	
			033.	

Hence, in the four cases respectively, the numbers are

$$\begin{aligned}
&\tfrac{1}{2}(k'^2 + 3k' + 2),\\
&\tfrac{1}{2}(k'^2 + k'),\\
&\tfrac{1}{2}(k'^2 + k'),\\
&\tfrac{1}{2}(k'^2 - k'),
\end{aligned}$$

giving a total

$$= \tfrac{1}{2}(4k'^2 + 4k' + 2),\ = \tfrac{1}{2}\{(2k'+1)^2 + 1\},$$

that is, $= \frac{1}{2}(k^2+1)$. And similarly the number is $= \frac{1}{2}(k^2+1)$ in the other three cases respectively.

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Preparation for the Transformation. Art. Nos. 22 to 44 (*several sub-headings*).

The Hermitian quartic matrix.

22. Observing that, for the adopted form of the Π- or Θ-functions, the periods

$$\begin{array}{lll} \omega_0,\ \upsilon_0 & \text{are} & 1,\ 0, \\ \omega_1,\ \upsilon_1 & & 0,\ 1, \\ \omega_2,\ \upsilon_2 & & A,\ H, \\ \omega_3,\ \upsilon_3 & & H,\ B, \end{array}$$

so that we have

$$\omega_0\upsilon_2 - \omega_2\upsilon_0 + \omega_1\upsilon_3 - \omega_3\upsilon_1, \quad = H - 0 + 0 - H, \ = 0,$$

we have to consider the automorphic transformation of the bilinear form

$$\omega_0\upsilon_2 - \omega_2\upsilon_0 + \omega_1\upsilon_3 - \omega_3\upsilon_1.$$

23. We write

$$(\omega_0,\ \omega_1,\ \omega_2,\ \omega_3) = \left(\begin{array}{llll} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right)\!\!\!\big\langle(\Omega_0,\ \Omega_1,\ \Omega_2,\ \Omega_3),$$

$$(\upsilon_0,\ \upsilon_1,\ \upsilon_2,\ \upsilon_3) = \left(\begin{array}{llll} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right)\!\!\!\big\langle(\Upsilon_0,\ \Upsilon_1,\ \Upsilon_2,\ \Upsilon_3),$$

and the coefficients are assumed to be such that we have identically

$$\omega_0\upsilon_2 - \omega_2\upsilon_0 + \omega_1\upsilon_3 - \omega_3\upsilon_1 = k\,(\Omega_0\Upsilon_2 - \Omega_2\Upsilon_0 + \Omega_1\Upsilon_3 - \Omega_3\Upsilon_1),$$

where k is in the sequel taken to be a positive integer. We obtain by direct substitution the value of $\omega_0\upsilon_2 - \omega_2\upsilon_0 + \omega_1\upsilon_3 - \omega_3\upsilon_1$ in the following form:

	Ω_0	Ω_1	Ω_2	Ω_3
Υ_0	$a_0c_0 - c_0a_0 + b_0d_0 - d_0b_0$	$a_1c_0 - c_1a_0 + b_1d_0 - d_1b_0$	$a_2c_0 - c_2a_0 + b_2d_0 - d_2b_0$	$a_3c_0 - c_3a_0 + b_3d_0 - d_3b_0$
Υ_1	$a_0c_1 - c_0a_1 + b_0d_1 - d_0b_1$	$a_1c_1 - c_1a_1 + b_1d_1 - d_1b_1$	$a_2c_1 - c_2a_1 + b_2d_1 - d_2b_1$	$a_3c_1 - c_3a_1 + b_3d_1 - d_3b_1$
Υ_2	$a_0c_2 - c_0a_2 + b_0d_2 - d_0b_2$	$a_1c_2 - c_1a_2 + b_1d_2 - d_1b_2$	$a_2c_2 - c_2a_2 + b_2d_2 - d_2b_2$	$a_3c_2 - c_3a_2 + b_3d_2 - d_3b_2$
Υ_3	$a_0c_3 - c_0a_3 + b_0d_3 - d_0b_3$	$a_1c_3 - c_1a_3 + b_1d_3 - d_1b_3$	$a_2c_3 - c_2a_3 + b_2d_3 - d_2b_3$	$a_3c_3 - c_3a_3 + b_3d_3 - d_3b_3$

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viz. equating this to its value $k(\Omega_0\Upsilon_2 - \Omega_2\Upsilon_0 + \Omega_1\Upsilon_3 - \Omega_3\Upsilon_1)$, we have 4 identities and 12 equations which are, in fact, 6 equations occurring each twice. We have thus six equations, which are the conditions in order that the matrix may be the matrix of automorphic transformation of the bilinear form $\omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1$. The six equations may be written

$$\begin{aligned}(ac+bd)_{01} &= 0,\\ (ac+bd)_{02} &= k,\\ (ac+bd)_{03} &= 0,\\ (ac+bd)_{12} &= 0,\\ (ac+bd)_{13} &= k,\\ (ac+bd)_{23} &= 0,\end{aligned}$$

viz. the first of these equations is $a_0c_1 - a_1c_0 + b_0d_1 - b_1d_0 = 0$, and so in other cases. It is convenient to remark that each interchange of two letters a and c, b and d, also interchange of the suffixes 0 and 2, produces a change of sign; thus the second equation may be written $(ca+db)_{02} = -k$, or $(ca+db)_{20} = k$.

24. The inverse matrix is found to be

$$\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right)^{-1} = \frac{1}{k}\left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2 \\ c_3, & d_3, & -a_3, & -b_3 \\ -c_0, & -d_0, & a_0, & b_0 \\ -c_1, & -d_1, & a_1, & b_1 \end{array}\right);$$

and the determinant of each of the matrices in this formula is $= k^2$.

We have thus

$$k(\Omega_0,\ \Omega_1,\ \Omega_2,\ \Omega_3) = \left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2 \\ c_3, & d_3, & -a_3, & -b_3 \\ -c_0, & -d_0, & a_0, & b_0 \\ -c_1, & -d_1, & a_1, & b_1 \end{array}\right)\!\!\big(\omega_0,\ \omega_1,\ \omega_2,\ \omega_3\big);$$

and the like formula for the Υ, v. Substituting these values in the equation

$$k(\Omega_0\Upsilon_2 - \Omega_2\Upsilon_0 + \Omega_1\Upsilon_3 - \Omega_3\Upsilon_1) = \omega_0 v_2 - \omega_2 v_0 + \omega_1 v_3 - \omega_3 v_1,$$

we obtain 6 new equations, which are in a different form the conditions for the automorphic transformation.

The 6 new equations may be written

$$\begin{aligned}(02+13)_{cd} &= 0,\\ (02+13)_{ac} &= k,\\ (02+13)_{bc} &= 0,\\ (02+13)_{ad} &= 0,\\ (02+13)_{bd} &= k,\\ (02+13)_{ab} &= 0,\end{aligned}$$

viz. these equations are

$$c_0d_2 - c_2d_0 + c_1d_3 - c_3d_1 = 0, \text{ \&c.}$$

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25. It is worth while to show how the foregoing formula for the inverse matrix comes out. Take for instance the diagonal minor

$$\left|\begin{array}{lll} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{array}\right|,$$

this is

$$= c_1\,(db)_{23} + c_2\,(db)_{31} + c_3\,(db)_{12},$$

which is

$$= c_1\,(ac)_{23} + c_2\,\{k + (ac)_{31}\} + c_3\,(ac)_{12},$$
$$= c_2 k,$$

since the remaining terms destroy each other. And dividing by the determinant, which is $= k^2$, we have the term $c_2 \div k$ of the inverse matrix.

The Symmetrical Hermitian Matrix.

26. We may consider a symmetrical Hermitian matrix, say the matrix

$$\left(\begin{array}{l|llll} & \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ & \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ & \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ & \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array}\right),$$

viz. we have

$$\begin{aligned} \mathfrak{AC} - \mathfrak{G}^2 \; &+ \mathfrak{HN} - \mathfrak{LF} &= \phi, \\ \mathfrak{HN} - \mathfrak{LF} \; &+ \mathfrak{BD} - \mathfrak{M}^2 &= \phi, \\ \mathfrak{AF} - \mathfrak{GH} \; &+ \mathfrak{HM} - \mathfrak{BL} &= 0, \\ \mathfrak{AN} - \mathfrak{LG} \; &+ \mathfrak{HD} - \mathfrak{LM} &= 0, \\ \mathfrak{CH} - \mathfrak{FG} \; &+ \mathfrak{BN} - \mathfrak{MF} &= 0, \\ \mathfrak{GN} - \mathfrak{CL} \; &+ \mathfrak{FD} - \mathfrak{MN} &= 0. \end{aligned}$$

The characteristic property is that, effecting a Hermitian transformation, we have a new symmetrical Hermitian matrix

$$\left(\begin{array}{l|llll} & \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ & \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ & \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ & \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array}\right)\!\!\big)\!\!\big(a_0x + a_1y + a_2z + a_3w,\; b_0x + \ldots,\; c_0x + \ldots,\; d_0x + \ldots)^2$$

$$= \left(\begin{array}{l|llll} & \mathfrak{A}', & \mathfrak{H}', & \mathfrak{G}', & \mathfrak{L}' \\ & \mathfrak{H}', & \mathfrak{B}', & \mathfrak{F}', & \mathfrak{M}' \\ & \mathfrak{G}', & \mathfrak{F}', & \mathfrak{C}', & \mathfrak{N}' \\ & \mathfrak{L}', & \mathfrak{M}', & \mathfrak{N}', & \mathfrak{D}' \end{array}\right)\!\!\big)\!\!\big(x, y, z, w)^2.$$

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In fact, the new matrix is

$$(\mathfrak{A}'\ldots) = \left(\begin{array}{cccc} a_0, & b_0, & c_0, & d_0 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{array}\right) (\mathfrak{A}\ldots) \left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right),$$

where the first factor, *quà* transposed matrix, is Hermitian. Hence the product is a symmetrical $k^2\phi$ matrix.

27. Write

$$\Delta_0 = H_0^2 - A_0B_0 - (\eta^2 - \alpha\beta),\quad \theta = \alpha\beta - \eta^2,$$
$$\delta\ = 2H_0\eta - \alpha B_0 - \beta A_0,$$

and consider the following matrix

$$\left(\begin{array}{cccc} \beta\ , & -\eta\ , & -\eta H_0 + \beta A_0, & -\eta B_0 + \beta H_0 \\ -\eta\ , & \alpha\ , & -\eta A_0 + \alpha H_0, & -\eta H_0 + \alpha B_0 \\ -\eta H_0 + \beta A_0, & -\eta A_0 + \alpha H_0, & +\alpha\Delta_0 - \delta A_0, & -\delta H_0 + \eta\Delta_0 \\ -\eta B_0 + \beta H_0, & -\eta H_0 + \alpha B_0, & -\delta H_0 + \eta\Delta_0, & +\beta\Delta_0 - \delta B_0 \end{array}\right)\!\between\! x, y, z, w)^2$$

$$= (\beta, -\eta, \alpha \between x + A_0z + H_0\omega_1y + H_0z + B_0w)^2 - (\eta^2 - \alpha\beta)(\alpha z^2 + 2\eta zw + \beta w^2);$$

it is easily shown that this is a symmetrical θ^2-matrix. In fact, representing it for a moment by

$$\left(\begin{array}{cccc} \mathfrak{A}, & \mathfrak{H}, & \mathfrak{G}, & \mathfrak{L} \\ \mathfrak{H}, & \mathfrak{B}, & \mathfrak{F}, & \mathfrak{M} \\ \mathfrak{G}, & \mathfrak{F}, & \mathfrak{C}, & \mathfrak{N} \\ \mathfrak{L}, & \mathfrak{M}, & \mathfrak{N}, & \mathfrak{D} \end{array}\right),$$

so that $\mathfrak{A} = \beta$, $\mathfrak{B} = \alpha$, $\mathfrak{H} = -\eta$, &c., we have

$$\mathfrak{A}\mathfrak{C} - \mathfrak{G}^2 + \mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} = \beta(\alpha\Delta_0 - \delta A_0) - (-\eta H_0 + \beta A_0)^2$$
$$- \eta(-\delta H_0 + \eta\Delta_0) - (-\eta B_0 + \beta H_0)(-\eta A_0 + \alpha H_0),$$

which is

$$= \alpha\beta\Delta_0 - \beta\delta A_0 - \eta^2H_0^2 + 2\beta\eta A_0H_0 - \beta^2A_0^2 + \eta\delta H_0 - \eta^2\Delta_0 - \eta^2A_0B_0 + \eta\alpha B_0H_0 + \eta\beta A_0H_0 - \alpha\beta H_0^2;$$

or, for the two terms $-\beta\delta A_0 + \eta\delta H_0$, $= (\eta H_0 - \beta A_0)\,\delta$, substituting the value

$$(\eta H_0 - \beta A_0)(2H_0\eta - \alpha B_0 - \beta A_0),$$

the whole is found to be

$$= (\alpha\beta - \eta^2)(\Delta_0 - H_0 + A_0B_0),\ \text{that is,}\ = (\alpha\beta - \eta^2)^2 = \theta^2.$$

And, similarly, $\mathfrak{H}\mathfrak{N} - \mathfrak{L}\mathfrak{F} + \mathfrak{B}\mathfrak{D} - \mathfrak{M}^2$ is found to be $= \theta^2$; and the remaining four combinations of terms to be each of them $= 0$; the matrix is thus a symmetrical θ^2-matrix.

28. It is to be added that the diagonal minors and the determinant

$$\mathfrak{A},\quad \mathfrak{A}\mathfrak{B} - \mathfrak{H}^2,\quad \mathfrak{A}\mathfrak{B}\mathfrak{C} + \&\text{c.},\quad \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} + \&\text{c.},$$

have respectively the values β, θ, $\alpha\theta^2$, θ^4; viz. if α, β, θ are positive, then these are all positive; or the last-mentioned quadric function is a definite positive form.

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The general matrix resumed: Arithmetical theory.

29. The matrix containing, as before, the parameter k, may be called a k-matrix; if k be $=1$, it is a unit-matrix. From the fundamental equation

$$k(\Omega_0 \Upsilon_2 - \Omega_2 \Upsilon_0 + \Omega_1 \Upsilon_3 - \Omega_3 \Upsilon_1) = \omega_0 \upsilon_2 - \omega_2 \upsilon_0 + \omega_1 \upsilon_3 - \omega_3 \upsilon_1,$$

it at once follows that, compounding a k-matrix with a k'-matrix, we have a kk'-matrix; and in particular, compounding a k-matrix with a unit-matrix, we have a k-matrix.

30. The symbols a_0, a_1, b_0, &c., and k, have thus far been arbitrary magnitudes, but we now take them to be integers; and we consider in particular the case where k is a positive odd prime. The number of k-matrices is of course infinite, but if we regard as equivalent any two such matrices which are derivable one from the other by post-multiplication by a unit-matrix (viz. U being a unit-matrix, the matrices M and $M.U$ are regarded as equivalent), then the number of distinct k-matrices is finite, and $=1+k+k^2+k^3$.

The first step is to show that we can by post-multiplication, by a properly determined unit-matrix, reduce the k-matrix to the form

$$\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ 0, & b_1, & b_2, & b_3 \\ 0, & 0, & c_2, & 0 \\ 0, & 0, & d_2, & d_3 \end{array}\right),$$

these values being such as to satisfy identically two out of the six conditions; the remaining conditions present themselves under the two equivalent forms

$$a_0c_2 = k,\ \ b_1d_3 = k,\ \ a_0b_2 + a_1b_3 - a_3b_1 = 0,\ \ a_0d_2 + a_1d_3 = 0,$$

and

$$a_0c_2 = k,\ \ b_1d_3 = k,\ \ a_1c_2 + b_1d_2 \qquad = 0,\ \ -a_3c_2 + b_2d_3 - b_3d_2 = 0.$$

Hence a_0, $c_2 = 1$, k, or k, 1; and b_1, $d_3 = 1$, k, or k, 1; so that, combining these pairs of values, we have four different types of matrix, each type depending on the coefficients a_1, d_2, a_2, b_2, a_3, b_3, connected together by two equations. But the forms of the same type are not distinct from each other, and we have to determine for each of the four types a system of non-equivalent forms comprised therein, and such that from these, by post-multiplication by a unit-matrix as before, the other forms of the type can be obtained. This final system is:—

$$\text{I.}\ \left|\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & k \end{array}\right|, \qquad \text{II.}\ \left|\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & k, & 0, & i \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right|,$$

$$\text{III.}\ \left|\begin{array}{cccc} k, & i, & i', & 0 \\ 0, & 1, & 1, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & -i, & k \end{array}\right|, \qquad \text{IV.}\ \left|\begin{array}{cccc} k, & 0, & i', & i \\ 0, & k, & i, & i'' \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{array}\right|,$$

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where i, i', i'' are integers, having each of them any one of the values 0, 1, 2,..., $k-1$, viz. there are in the four types 1, k, k^2, k^3 forms respectively, and the number of forms is thus $=1+k+k^2+k^3$, as already mentioned. I abstain from the further details of the proof.

There is obviously a like theory for which, in place of post-multiplication, we have pre-multiplication; viz. here the matrices M and $U.M$ are regarded as equivalent.

31. Any two k-matrices are reducible one to the other by a combined pre- and post-multiplication; viz. we have always $M'=U.M.U'$, where M, M' are any two k-matrices, and U, U' properly determined unit-matrices; and in particular, M' being any given k-matrix, this is expressible in the foregoing form, where M denotes the principal matrix

$$\left(\begin{array}{cccc} 1, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & k, & 0 \\ 0, & 0, & 0, & k \end{array}\right).$$

Congruence theorems, k an odd number.

32. Taking k an odd number, and using throughout = instead of $\equiv$ (mod. 2), we have the following congruences:

$$(a_0a_2+a_1a_3,\ b_0b_2+b_1b_3,\ c_0c_2+c_1c_3,\ d_0d_2+d_1d_3)$$

$$=\left(\begin{array}{cccc} a_0, & a_1, & a_2, & a_3 \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}\right)\!\!\!\big\backslash\!\!\!\big\backslash (a_2c_2+b_2d_2,\ a_3c_3+b_3d_3,\ a_0c_0+b_0d_0,\ a_1c_1+b_1d_1);$$

or, conversely,

$$(a_2c_2+b_2d_2,\ a_3c_3+b_3d_3,\ a_0c_0+b_0d_0,\ a_1c_1+b_1d_1)$$

$$=\left(\begin{array}{cccc} c_2, & d_2, & -a_2, & -b_2 \\ c_3, & d_3, & -a_3, & -b_3 \\ -c_0, & -d_0, & a_0, & b_0 \\ -c_1, & -d_1, & a_1, & b_1 \end{array}\right)\!\!\!\big\backslash\!\!\!\big\backslash (a_0a_2+a_1a_3,\ b_0b_2+b_1b_3,\ c_0c_2+c_1c_3,\ d_0d_2+d_1d_3),$$

where observe that on the right-hand side the signs $-$ may be changed into $+$; in fact, to the modulus 2, we have for any integer value whatever $-p\equiv+p$.

33. The first congruence is

$$a_0a_2+a_1a_3=a_0(a_2c_2+b_2d_2)+a_1(a_3c_3+b_3d_3)+a_2(a_0c_0+b_0d_0)+a_3(a_1c_1+b_1d_1),\ \text{say}\ X=Y,$$

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and to verify this, I see no other method than that of considering separately all the combinations of even and odd values of a_0, a_1, a_2, a_3; viz. we have the 16 cases

Case	a_0,	a_2,	a_1,	a_3	X,	Y
1	0	0	0	0	does not exist	
2	0	0	0	1	0	0
3	0	0	1	0	0	0
4	0	0	1	1	1	1
5	0	1	0	0	0	0
6	0	1	0	1	0	0
7	0	1	1	0	0	0
8	0	1	1	1	1	1
9	1	0	0	0	0	0
10	1	0	0	1	0	0
11	1	0	1	0	0	0
12	1	0	1	1	1	1
13	1	1	0	0	1	1
14	1	1	0	1	1	1
15	1	1	1	0	1	1
16	1	1	1	1	0	0

viz. the X column gives in each case the value of X, $= a_0a_2 + a_1a_3$, and then it has to be shown that Y has the same value. The coefficients satisfy the conditions

$$\begin{aligned} a_0c_2 - a_2c_0 + a_1c_3 - a_3c_1 &= 1, \\ b_0d_2 - b_2d_0 + b_1d_3 - b_3d_1 &= 1, \\ a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1 &= 0, \\ a_0d_2 - a_2d_0 + a_1d_3 - a_3d_1 &= 0, \\ b_0c_2 - b_2c_0 + b_1c_3 - b_3c_1 &= 0, \\ c_0d_2 - c_2d_0 + c_1d_3 - c_3d_1 &= 0, \end{aligned}$$

so that from the first equation we cannot have a_0, a_1, a_2, a_3 each $= 0$, or the first case does not exist. As to the remaining cases, it is easy to see that they group themselves as follows: 2, 3, 5, 9: 4, 13: 6, 7, 10, 11: 8, 12, 14, 15: 16: the proof being substantially the same for the several cases in the same group.

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34. Case 2. We have $Y = b_1 d_1$ which should be $=0$, and in fact, the six equations give $b_1 = 0$, $d_1 = 0$, whence $Y = 0$.

Case 4. We have $Y = c_1 + c_3 + b_1 d_1 + b_3 d_3$ which should $=1$, and in fact, the six equations give $c_3 - c_1 = 1$, $b_3 - b_1 = 0$, $d_3 - d_1 = 0$; whence $c_1 + c_3 = 1$, $b_1 d_1 + b_3 d_3 = 0$, and therefore $Y = 1$.

Case 6. Here $Y = b_0 d_0 + b_1 d_1$ which should $=0$; and in fact, the six equations give $b_0 + b_1 = 0$, $d_0 + d_1 = 0$, whence $Y = 0$.

Case 16. Here $Y = c_0 + c_1 + c_2 + c_3 + b_0 d_0 + b_1 d_1 + b_2 d_2 + b_3 d_3$, which should be $=0$; the six equations give

$$-c_0 - c_1 + c_2 + c_3 = 1, \; -b_0 - b_1 + b_2 + b_3 = 0,$$

$$-d_0 - d_1 + d_2 + d_3 = 0, \text{ and } b_0 d_2 - b_2 d_0 + b_1 d_3 - b_3 d_1 = 1.$$

Writing

$$b_0 + b_2 = b_1 + b_3 \text{ and } d_0 + d_2 = d_1 + d_3,$$

we find

$$b_0 d_0 + b_2 d_2 + b_0 d_2 + b_2 d_0 = b_1 d_1 + b_3 d_3 + b_1 d_3 + b_3 d_1,$$

that is,

$$b_0 d_0 + b_1 d_1 + b_2 d_2 + b_3 d_3 = b_0 d_2 + b_2 d_0 + b_1 d_3 + b_3 d_1, \; =1;$$

and $c_0 + c_1 + c_2 + c_3 = 1$, whence $Y = 0$.

35. Case 8. This is the only case of any difficulty: we have

$$Y = c_1 + c_3 + b_0 d_0 + b_1 d_1 + b_3 d_3,$$

which should be $=1$. The six equations give

$$-c_0 + c_3 - c_1 = 1, \; -b_0 + b_3 - b_1 = 0, \; -d_0 + d_3 - d_1 = 0,$$

or, say

$$c_0 = -1 - c_1 + c_3, \; b_0 = -b_1 + b_3, \; d_0 = -d_1 + d_3;$$

or, substituting these values and omitting even terms

$$Y = c_1 + c_3 - b_1 d_3 - b_3 d_1.$$

The remaining three of the six equations are

$$b_0 d_2 - b_2 d_0 + b_1 d_3 - b_3 d_1 = 1, \; b_0 c_2 - b_2 c_0 + b_1 c_3 - b_3 c_1 = 0, \; c_0 d_2 - c_2 d_0 + c_1 d_3 - c_3 d_1 = 0;$$

or, substituting for b_0, c_0, d_0 their values, these become

$$\begin{aligned}
(1 + c_1 - c_3)\, b_2 + (b_3 - b_1)\, c_2 \qquad\qquad\qquad\qquad &= -b_1 c_3 + b_3 c_1, \\
(d_1 - d_3)\, b_2 + \qquad\qquad\qquad (b_3 - b_1)\, d_2 &= 1 - b_1 d_3 + b_3 d_1, \\
(d_1 - d_3)\, c_2 + (-1 - c_1 + c_3)\, d_2 &= -c_1 d_3 + c_3 d_1;
\end{aligned}$$

we can from these equations eliminate b_2, c_2, d_2; viz. from the first and third equations eliminating c_2, we have

$$(1 + c_1 - c_3)\{(d_3 - d_1)\, b_2 + (b_1 - b_3)\, d_2\} = (d_3 - d_1)(-b_1 c_3 + b_3 c_1) + (b_3 - b_1)(-c_1 d_3 + c_3 d_1),$$

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and this, by means of the second equation, becomes

$$(1 + c_1 - c_3)(-1 + b_1 d_3 - b_3 d_1) = (d_3 - d_1)(-b_1 c_3 + b_3 c_1) + (b_3 - b_1)(-c_1 d_3 + c_3 d_1),$$

viz. reducing, this is

$$-c_1 + c_3 - 1 + b_1 d_3 - b_3 d_1 = 0;$$

and in virtue of it we have $Y = c_1 + c_3 - b_1 d_3 - b_3 d_1 = 1$.

It can be further shown that, to the modulus 2 (k being, as before, odd), we have

$$(a_0 a_2 + a_1 a_3)(c_0 c_2 + c_1 c_3) + (b_0 b_2 + b_1 b_3)(d_0 d_2 + d_1 d_3) = 0,$$
$$(a_0 c_0 + b_0 d_0)(a_2 c_2 + b_2 d_2) + (a_1 c_1 + b_1 d_1)(a_3 c_3 + b_3 d_3) = 0.$$

To prove the first equation, write for a moment

$$\Omega = (a_0 a_2 + a_1 a_3)(c_0 c_2 + c_1 c_3), \quad \Omega' = (b_0 b_2 + b_1 b_3)(d_0 d_2 + d_1 d_3),$$
$$X = (a_0 c_1 - a_1 c_0)(a_2 c_3 - a_3 c_2), \quad X' = (b_0 d_1 - b_1 d_0)(b_2 d_3 - b_3 d_2);$$

then in virtue of the equations

$$a_0 c_1 - a_1 c_0 + b_0 d_1 - b_1 d_0,$$
$$a_2 c_3 - a_3 c_2 + b_2 d_3 - b_3 d_2,$$

we have $X = X'$. But we have identically

$$\Omega - X = (a_0 c_2 + a_1 c_3)(a_2 c_0 + a_3 c_1),$$

and from the equation

$$a_0 c_2 - a_2 c_0 + a_1 c_3 - a_3 c_1 = 1,$$

that is, $a_2 c_0 + a_3 c_1 = -1 + a_0 c_2 + a_1 c_3$, we have $\Omega - X = 0$; and similarly $\Omega' - X' = 0$, that is, $\Omega - \Omega' = X - X' = 0$; we have thus the required equation $\Omega + \Omega' = 0$. In a similar manner the second equation may be verified.

36. Write

$$\mu' = \mu a_0 + \nu a_1 + q a_2 + r a_3 + a_0 a_2 + a_1 a_3,$$
$$\nu' = \mu b_0 + \nu b_1 + q b_2 + r b_3 + b_0 b_2 + b_1 b_3,$$
$$q' = \mu c_0 + \nu c_1 + q c_2 + r c_3 + c_0 c_2 + c_1 c_3,$$
$$r' = \mu d_0 + \nu d_1 + q d_2 + r d_3 + d_0 d_2 + d_1 d_3.$$

It is to be shown that to the modulus 2 we have

$$\mu' q' + \nu' r' = \mu q + \nu r.$$

In fact, forming the value of $\mu' q' + \nu' r'$, we have first a constant term (term without μ, ν, q, r) which vanishes; next writing $\mu^2 = \mu$, the whole term in μ is

$$a_0 c_0 + b_0 d_0 + a_0 (c_0 c_2 + c_1 c_3) + b_0 (d_0 d_2 + d_1 d_3) + c_0 (a_0 a_2 + a_1 a_3) + d_0 (b_0 b_2 + b_1 b_3),$$

which also vanishes; and similarly the terms in ν, q, r each of them vanish; there remain only the terms in $\mu\nu$, μq, &c. The coefficient of μq is $a_0 c_2 + a_2 c_0 + b_0 d_2 + b_2 d_0$,

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which (always to the modulus 2) is $= a_0c_2 - a_2c_0 + b_0d_2 - b_2d_0$, that is, it is $=1$; and similarly the coefficient of νr is $=1$; and in like manner the coefficients of the other terms are each $=0$; we have thus the required congruence $\mu'q' + \nu'r' = \mu q + \nu r$.

The quintic matrix.

37. I consider a quintic matrix composed of the foregoing coefficients $(a, b, c, d)_{0,1,2,3}$, viz. the matrix contained in a linear transformation which I write as follows:

$$\begin{array}{r|ccccc|l} & 01 & 21 & 2.02 & 03 & 32 & \\ \hline (T', P', Q', R', S') = ab & . & . & . & . & . & (T, P, Q, R, S), \\ cb & . & . & . & . & . & \\ ac & . & . & -k & . & . & \\ ad & . & . & . & . & . & \\ dc & . & . & . & . & . & \end{array}$$

read

$$T' = (ab)_{01}\, T + (ab)_{21}\, P + 2\,(ab)_{02}\, Q + (ab)_{03}\, R + (ab)_{32}\, S,$$

where as before $(ab)_{01} = a_0b_1 - a_1b_0$, &c., and so in other cases; in particular, observe that, in the expression of Q', the term involving Q is $\{-k + 2\,(ac)_{02}\}\,Q$ which, in virtue of the relation $(ac + bd)_{02} = k$, may also be written $\{(ac)_{02} - (bd)_{02}\}\,Q$. I notice that in Hermite's paper, p. 366, the term is in effect written without the $-k$, $= 2\,(ac)_{02}\,Q$; the correction of this erratum and of a corresponding one, p. 366, is made p. 787 at the conclusion of the memoir.

38. The matrix is automorphic for the form $T^2 - PR - TS$, viz. we have identically

$$Q'^2 - P'R' - T'S' = k^2\,(Q^2 - PR - TS).$$

As a partial verification, consider in $Q'^2 - P'R' - T'S'$ the term containing Q^2. The coefficient of Q^2 is

$$\{-k + 2\,(ac)_{02}\}^2 - 4\,(cb)_{02}\,(ad)_{02} - 4\,(ab)_{02}\,(dc)_{02},$$

where the first term is

$$\{(ac)_{02} - (bd)_{02}\}^2, \text{ which is } = \{(ac)_{02} + (bd)_{02}\}^2 - 4\,(ac)_{02}\,(bd)_{02} = k^2 - 4\,(ac)_{02}\,(bd)_{02}.$$

Hence, observing that we have

$$(ac)_{02}\,(bd)_{02} + (cb)_{02}\,(ad)_{02} + (ab)_{02}\,(dc)_{02} = 0,$$

the whole coefficient is $=k^2$, as it should be; and in like manner the verification may be effected for any other term.

39. We require the following formulæ:

$$\left(\begin{array}{ccc} -c_0, & a_0, & b_0 \\ -c_1, & a_1, & b_1 \\ -c_2, & a_2, & b_2 \\ -c_3, & a_3, & b_3 \end{array}\right)(T', P', Q') = k\left(\begin{array}{ccccc} ., & b_1, & -b_0, & ., & -b_3 \\ ., & ., & b_1, & -b_0, & -b_2 \\ -b_1, & ., & -b_2, & -b_3, & . \\ b_0, & b_2, & b_3, & ., & . \end{array}\right)(T, P, Q, R, S),$$

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that is,

$$-c_0T' + a_0P' + b_0Q' = k(b_1P - b_0Q - b_3S), \text{ \&c.},$$

and

$$\left(\begin{array}{ccc} -d_0, & a_0, & b_0 \\ -d_1, & a_1, & b_1 \\ -d_2, & a_2, & b_2 \\ -d_3, & a_3, & b_3 \end{array}\right)(T', Q', R') = k\left(\begin{array}{ccccc} ., & -a_1, & a_0, & ., & a_3 \\ ., & ., & -a_1, & a_0, & -a_2 \\ a_1, & ., & a_2, & a_3, & . \\ -a_0, & -a_2, & -a_3, & ., & . \end{array}\right)(T, P, Q, R, S).$$

From the first set, multiplying the first, third and fourth equations by T, P, Q respectively and adding, and again multiplying the second, third and fourth equations by T, Q, R, and adding, we obtain

$$-(c_0T + c_2P + c_3Q)\,T' + (a_0T + a_2P + a_3Q)\,P' + (b_0T + b_2P + b_3Q)\,Q' = kb_3(Q^2 - PR - TS),$$
$$-(c_1T + c_2Q + c_3R)\,T' + (a_1T + a_2Q + a_3R)\,P' + (b_1T + b_2Q + b_3R)\,Q' = -kb_2(Q^2 - PR - TS);$$

and similarly, from the second set of equations, we obtain

$$-(d_0T + d_2P + d_3Q)\,T' + (a_0T + a_2P + a_3Q)\,Q' + (b_0T + b_2P + b_3Q)\,R' = -ka_3(Q^2 - PR - TS),$$
$$-(d_1T + d_2Q + d_3R)\,T' + (a_1T + a_2Q + a_3R)\,Q' + (b_1T + b_2Q + b_3R)\,R' = ka_2(Q^2 - PR - TS).$$

40. We have the inverse system

$$\begin{array}{r|ccccc|l} & dc & ad & 2bd & cb & ab & \\ \hline (T, P, Q, R, S) = 32 & . & . & . & . & . & (T', P', Q', R', S'), \\ 03 & . & . & . & . & . & \\ 13 & . & . & -k & . & . & \\ 21 & . & . & . & . & . & \\ 01 & . & . & . & . & . & \end{array}$$

read

$$T = (dc)_{32}\,T' + (ad)_{32}\,P' + 2\,(bd)_{32}\,Q' + (cb)_{32}\,R' + (ab)_{32}\,S';$$

and in particular observe that, in the expression for Q, the term containing Q' is $\{-k + 2\,(bd)_{13}\}\,Q'$, where, in virtue of $(ac + bd)_{13} = k$, the coefficient of Q' is also $= (bd)_{13} - (ac)_{13}$.

41. These equations give

$$\left(\begin{array}{ccc} a_0, & a_2, & a_3 \\ b_0, & b_2, & b_3 \\ c_0, & c_2, & c_3 \\ d_0, & d_2, & d_3 \end{array}\right)(T, P, Q) = k\left(\begin{array}{ccccc} d_3, & ., & -a_3, & -b_3, & . \\ -c_3, & a_3, & b_3, & ., & . \\ ., & d_3, & -c_3, & ., & b_3 \\ ., & ., & d_3, & -c_3, & -a_3 \end{array}\right)(T', P', Q', R', S'),$$

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and

$$\left(\begin{array}{ccc} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{array}\right)\!\!\!\big\rangle\!\!\big(T, Q, R) = k \left(\begin{array}{ccccc} -d_2, & ., & a_2, & b_2, & . \\ c_2, & -a_2, & -b_2, & ., & . \\ ., & -d_2, & c_2, & ., & -b_2 \\ ., & ., & -d_2, & c_2, & a_2 \end{array}\right)\!\!\!\big\rangle\!\!\big(T', P', Q', R', S').$$

From the first set of equations, multiplying the first, third and fourth equations by $-S'$, $-R'$, $+Q'$ respectively, and adding, and again multiplying the second, third and fourth equations by $-S'$, Q', $-P'$ respectively, and adding, we deduce

$$T(-a_0S' - c_0R' + d_0Q') + P(-a_2S' - c_2R' + d_2Q') + Q(-a_3S' - c_3R' + d_3Q') = -kc_3(Q'^2 - P'R' - T'S'),$$

$$T(-b_0S' + c_0Q' - d_0P') + P(-b_2S' + c_2Q' - d_2P') + Q(-b_3S' + c_3Q' - d_3P') = kd_3(Q'^2 - P'R' - T'S');$$

and in like manner, from the second set of equations,

$$T(-a_1S' - c_1R' + d_1Q') + Q(-a_2S' - c_2R' + d_2Q') + R(-a_3S' - c_3R' + d_3Q') = -kd_2(Q'^2 - P'R' - T'S'),$$

$$T(-b_1S' + c_1Q' - d_1P') + Q(-b_2S' + c_2Q' - d_2P') + R(-b_3S' + c_3Q' - d_3P') = kc_2(Q'^2 - P'R' - T'S').$$

42. Assume that T, P, Q, R, S and T', P', Q', R', S' are linearly connected as above; and write

$$T : P : Q : R : S = 1 : A : H : B : H^2 - AB,$$

$$T' : P' : Q' : R' : S' = 1 : A' : H' : B' : H'^2 - A'B',$$

equations which establish a like relation between 1, A, H, B, $H^2 - AB$, and 1, A', H', B', $H'^2 - A'B'$. Observe that these forms are admissible since, if

$$T^2 - PR - QS = 0,$$

then also

$$Q'^2 - P'R' - T'S' = 0.$$

The quantities A, H, B were taken as the parameters of a theta-function; viz. taking

$$A, H, B = A_0 + i\alpha,\ H_0 + i\eta,\ B_0 + i\beta,$$

then $(\alpha, \eta, \beta \between x, y)^2$ must be a positive form (or what is the same thing, α and $\alpha\beta - \eta^2$ must be positive). If A', H', B' are also the parameters of a theta-function, then writing

$$A', H', B' = A_0' + i\alpha',\ H_0' + i\eta',\ B_0' + i\beta',$$

$(\alpha', \eta', \beta' \between x, y)^2$ must also be a positive form. It can be shown that, A', H', B' being determined as above, the former condition implies the latter one; viz. if the form $(\alpha, \eta, \beta \between x, y)^2$ be positive, then also the form $(\alpha', \eta', \beta' \between x, y)^2$ will be positive.

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Write

$$\begin{aligned} A &= A_0 + i\alpha, \quad A' = A_0' + i\alpha', \\ B &= B_0 + i\beta, \quad B' = B_0' + i\beta', \\ H &= H_0 + i\eta, \quad H' = H_0' + i\eta'. \\ \Delta &= \Delta_0 + i\delta, \\ \Delta_0 &= H_0^2 - A_0 B_0 - (\eta^2 - \alpha\beta), \\ \delta &= 2H_0\eta - B_0\alpha - A_0\beta, \end{aligned}$$

and let the quintic matrix in the foregoing transformation

$$(T', P', Q', R', S') = (M \between T, P, Q, R, S)$$

be represented by

	t	p	q	r	s
0					
1					
2					
3					
4					

43. It is to be shown that $\alpha' x^2 + 2\eta' xy + \beta' y^2$ is definite and positive.

We require α', β', η'; we have

$$A_0' + i\alpha' = \frac{(t_1, p_1, q_1, r_1, s_1)(1, A_0 + i\alpha, H_0 + i\eta, B_0 + i\beta, \Delta_0 + i\delta)}{(t_0, p_0, q_0, r_0, s_0)(\qquad\qquad ,, \qquad\qquad)} = \frac{M_1 + N_1 i}{M_0 + N_0 i},$$

and thence

$$\alpha' = \frac{M_0 N_1 - M_1 N_0}{M_0^2 + N_0^2} = \frac{1}{\square}(M_0 N_1 - M_1 N_0).$$

Now

$$\begin{aligned} M_0 N_1 - M_1 N_0 = \; & (t_0 p_1 - t_1 p_0)\, \alpha \\ & + (t_0 q_1 - t_1 q_0)\, \eta \\ & + (t_0 r_1 - t_1 r_0)\, \beta \\ & + (t_0 s_1 - t_1 s_0)\, \delta \\ & + (p_0 q_1 - p_1 q_0)(A_0 \eta - H_0 \alpha) \\ & + (p_0 r_1 - p_1 r_0)(A_0 \beta - B_0 \alpha) \\ & + (p_0 s_1 - p_1 s_0)(A_0 \delta - \Delta_0 \alpha) \\ & + (q_0 r_1 - q_1 r_0)(H_0 \beta - B_0 \eta) \\ & + (q_0 r_1 - q_1 r_0)(H_0 \beta - B_0 \eta) \\ & + (q_0 s_1 - q_1 s_0)(H_0 \delta - \Delta_0 \eta) \\ & + (r_0 s_1 - r_1 s_0)(B_0 \delta - \Delta_0 \beta); \end{aligned}$$

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and so for η', β' with only the change of t_1, p_1, q_1, r_1, s_1 into t_2, p_2, q_2, r_2, s_2 and t_3, p_3, q_3, r_3, s_3 respectively; we find, omitting a factor $\frac{k}{\square}$ throughout,

	$\alpha' =$	$\eta' =$	$\beta' =$
α	b_1^2,	$-a_1b_1$,	a_1^2,
η	$-2b_0b_1$,	$a_0b_1 + a_1b_0$,	$-2a_0a_1$,
β	b_0^2,	$-a_0b_0$,	a_0^2,
δ	$-(b_0b_2 + b_1b_3)$,	$\frac{1}{2}(a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1)$,	$-(a_0a_2 + a_1a_3)$,
$A_0\eta - H_0\alpha$	$-2b_1b_2$,	$a_1b_2 + a_2b_1$,	$-2a_1a_2$,
$H_0\beta - B_0\eta$	$+2b_0b_3$,	$-(a_0b_3 + a_3b_0)$,	$+2a_0a_3$,
$A_0\beta - B_0\alpha$	$b_0b_2 - b_1b_3$,	$-\frac{1}{2}(a_0b_2 + a_2b_0 - a_1b_3 - a_3b_1)$,	$a_0a_2 - a_1a_3$,
$A_0\delta - \Delta_0\alpha$	$-b_2^2$,	a_2b_2,	$-a_2^2$,
$H_0\delta - \Delta_0\eta$	$-2b_2b_3$,	$+a_2b_3 + a_3b_2$,	$-2a_2a_3$,
$B_0\delta - \Delta_0\beta$	$-b_3^2$,	a_3b_3,	$-a_3^2$,

and hence

$$\alpha' x^2 + 2\eta' xy + \beta' y^2$$

$$= \frac{k}{\square}\left(\begin{array}{cccc} \beta, & -\eta, & -\eta H_0 + \beta A_0, & -\eta B_0 + \beta H_0 \\ -\eta, & \alpha, & -\eta A_0 + \alpha H_0, & -\eta H_0 + \alpha B_0 \\ -\eta H_0 + \beta A_0, & -\eta A_0 + \alpha H_0, & +\alpha\Delta_0 - \delta A_0, & -\delta H_0 + \eta\Delta_0 \\ -\eta B_0 + \beta H_0, & -\eta H_0 + \alpha B_0, & \delta H_0 - \eta\Delta_0, & +\beta\Delta_0 - \delta B_0 \end{array}\right)(b_0x - a_0y,\ b_1x - a_1y,\ b_2x - a_2y,\ b_3x - a_3y)^2.$$

The right-hand is here definite and positive (*suprà* No. 28), hence also the left-hand is definite and positive.

44. As a specimen of the work, observe that we have

$$\begin{aligned} t_0p_1 - t_1p_0 &= (ab)_{01}(cb)_{21} - (ab)_{21}(cb)_{01} \\ &= b_1\{b_0(ac)_{21} + b_2(ac)_{10} + b_1(ac)_{02}\} \\ &= b_1[-b_0(bd)_{21} - b_2(bd)_{10} + b_1\{-(bd)_{02} + k\}] \\ &= b_1^2 k, \end{aligned}$$

since the remaining terms destroy each other. Dividing by $\square$, and then omitting the factor $\frac{k}{\square}$, we have thus the term αb_1^2 in the foregoing expression for α'.

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45. Consider $(a, b, c, d)_{0, 1, 2, 3}$, the components of a quartic matrix as above, and also T, P, Q, R, S; T', P', Q', R', S' connected as already mentioned; and write for shortness

$$z_0 = a_0x + b_0y, \quad z_1 = a_1x + b_1y,$$
$$z_2 = a_2x + b_2y, \quad z_3 = a_3x + b_3y,$$

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and consider the function

$$\Pi(x, y) = \Theta(z_0 + Az_2 + Hz_3,\ z_1 + Hz_2 + Bz_3) \exp. i\pi\{(z_0z_2 + z_1z_3) + (A, H, B\between z_2, z_3)^2\},$$

where Θ denotes the function

$$\Theta\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(A, H, B).$$

It is to be shown that Π is a function

$$\Pi\begin{pmatrix}\mu', & \nu'\\ q', & r'\end{pmatrix}(x, y)(A', H', B')(K, \text{def.}),$$

where the new parameters A', H', B' are given in terms of the original parameters A, H, B by the equations

$$1 : A : H : B : H^2 - AB = T : P : Q : R : S,$$
$$1 : A' : H' : B' : H'^2 - A'B' = T' : P' : Q' : R' : S',$$

and where μ', ν', q', r' have the values

$$\mu' = \mu a_0 + \nu a_1 + qa_2 + ra_3 + a_0a_2 + a_1a_3,$$
$$\nu' = \mu b_0 + \nu b_1 + qb_2 + rb_3 + b_0b_2 + b_1b_3,$$
$$q' = \mu c_0 + \nu c_1 + qc_2 + rc_3 + c_0c_2 + c_1c_3,$$
$$r' = \mu d_0 + \nu d_1 + qd_2 + rd_3 + d_0d_2 + d_1d_3;$$

viz. it is to be shown that the function Π, as above defined, satisfies the fundamental equations

$$\Pi(x+1, y) = (-)^{\mu'}\Pi(x, y),$$
$$\Pi(x, y+1) = (-)^{\nu'}\Pi(x, y),$$
$$\Pi(x + A', y + H') = (-)^{q'}\Pi(x, y) \exp. - 2\pi k(2x + A'),$$
$$\Pi(x + H', y + B') = (-)^{r'}\Pi(x, y) \exp. - 2\pi k(2y + B'),$$

and

$$\Pi(-x, -y) = (-)^{\mu'\nu' + q'r'}\Pi(x, y).$$

46. It is proper in the first place to show how it is that A', H', B' are capable of being the parameters of the new function.

Write for shortness

$$X = z_0 + Az_2 + Hz_3,$$
$$Y = z_1 + Hz_2 + Bz_3,$$

and suppose x changed into $x+1$ we have z_0, z_1, z_2, z_3 increased by a_0, a_1, a_2, a_3 respectively; and thence X, Y increased by $a_0 + Aa_2 + Ha_3$, $a_1 + Ha_2 + Ba_3$; the theta-function is thus changed into

$$\Theta(X + a_0 + Aa_2 + Ha_3,\quad Y + a_1 + Ha_2 + Ba_3),$$

viz. the arguments are increased by multiples of the quarter-periods (1, 0, A, H) and (0, 1, H, B).

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And, similarly, by the change of y into $y+1$, the theta-function is changed into

$$\Theta(X+b_0+Ab_2+Hb_3,\quad Y+b_1+Hb_2+Bb_3),$$

or the arguments are increased by integer multiples of the quarter-periods.

47. But suppose next that x, y are changed into $x+A'$, $y+B'$: we have z_0, z_1, z_2, z_3 increased by

$$A'a_0+H'b_0,\quad A'a_1+H'b_1,\quad A'a_2+H'b_2,\quad A'a_3+H'b_3,$$

and thence

$$\begin{aligned} &X \text{ increased by } A'(a_0+Aa_2+Ha_3)+H'(b_0+Ab_2+Hb_3),\ =c_0+Ac_2+Hc_3,\\ &Y \quad\text{,,}\quad\text{,,}\quad A'(a_1+Ha_2+Ba_3)+H'(b_1+Hb_2+Bb_3),\ =c_1+Hc_2+Bc_3,\end{aligned}$$

since these equalities are the before-mentioned equations

$$\begin{aligned} P'(Ta_0+Pa_2+Qa_3)+Q'(Tb_0+Pb_2+Qb_3)-T'(Tc_0+Pc_2+Qc_3)&=0,\\ P'(Ta_1+Qa_2+Ra_3)+Q'(Tb_1+Qb_2+Rb_3)-T'(Tc_1+Qc_2+Rc_3)&=0.\end{aligned}$$

It thus appears that, by the change of x, y into $x+A'$, $y+H'$, the theta-function is changed into

$$\Theta(X+c_0+Ac_2+Hc_3,\quad Y+c_1+Hc_2+Bc_3),$$

viz. we have again the arguments increased by integer multiples of the quarter-periods; and in like manner, by the change of x, y into $x+H'$, $y+B'$, the theta-function is changed into

$$\Theta(X+d_0+Ad_2+Hd_3,\quad Y+d_1+Hd_2+Bd_3),$$

or the arguments are again increased by integer multiples of the quarter-periods.

48. We have to complete the verifications, first for the equation

$$\Pi(x+1,\ y)=(-)^{\mu'}\,\Pi(x,\ y).$$

Reverting to the definition of Π, we have

$$\begin{aligned} &(-)^{\mu'}\,\Pi(x+1,\ y)\div\Pi(x,\ y)\\ &=(-)^{-\mu a_0-\nu a_1-qa_2-ra_3}\exp.-i\pi\{a_0a_2+a_1a_3\};\end{aligned}$$

$$\begin{aligned} \Theta(X+a_0+Aa_2+Ha_3,\ Y+a_1+Ha_2+Ba_3)&\div\Theta(X,\ Y)\\ =\exp.i\pi\{a_0z_2+a_2z_0+a_1z_3+a_3z_1+a_0x_2+a_1a_3\}&\times\\ \exp.i\pi\{2(Aa_2+Ha_3)z_2+2(Ha_2+Ba_3)z_3+(A,\ H,\ B)(a_2,\ a_3)^2\};\end{aligned}$$

the second line of this is

$$=(-)^{\mu a_0+\nu a_1+qa_2+ra_3}\exp.i\pi\{-2a_2X-2a_3Y-(A,\ H,\ B)(a_2,\ a_3)^2\},$$

and the right-hand side of the equation will thus be $=1$ if only the whole argument of the exponential be $=0$; that is, omitting the terms which destroy each other and the common factor $i\pi$, if only

$$\begin{aligned} &-2a_2X-2a_3Y\\ &+a_0z_2+a_2z_0+a_1z_3+a_3z_1\\ &+2(Aa_2+Ha_3)z_2+2(Ha_2+Ba_3)z_3=0.\end{aligned}$$

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Substituting herein for X, Y their values

$$a_0 + Aa_2 + Ha_3, \quad a_1 + Ha_2 + Ba_3,$$

the equation becomes

$$a_0 z_2 - a_2 z_0 + a_1 z_3 - a_3 z_1 = 0,$$

and finally substituting herein for z_0, z_1, z_2, z_3 their values, the coefficient of x is identically $= 0$, and the coefficient of y is

$$a_0 b_2 - a_2 b_0 + a_1 b_3 - a_3 b_1,$$

which is $(02 + 13)_{ab}$, and is thus $= 0$. This completes the proof of the equation

$$\Pi(x+1, y) = (-)^{\mu'} \Pi(x, y);$$

and we have of course a precisely similar proof for the equation

$$\Pi(x, y+1) = (-)^{\nu'} \Pi(x, y).$$

49. For the next equation, writing for shortness α_0, α_1, α_2, α_3 for $A'a_0 + H'b_0$, $A'a_1 + H'b_1$, $A'a_2 + H'b_2$, $A'a_3 + H'b_3$, so that z_0, z_1, z_2, z_3 are increased by α_0, α_1, α_2, α_3 respectively, we have

$$(-)^{-q'} \Pi(x + A', y + H') \div \Pi(x, y)$$
$$= (-)^{-\mu c_0 - \nu c_1 - q c_2 - r c_3} \exp. -i\pi (c_0 c_2 + c_1 c_3);$$

$$\Theta(X + c_0 + Ac_2 + Hc_3, Y + c_1 + Hc_2 + Bc_3) \div \Theta(x, y)$$
$$= \exp. i\pi \{\alpha_0 z_2 + \alpha_2 z_0 + \alpha_1 z_3 + \alpha_3 z_1 + \alpha_0 \alpha_2 + \alpha_1 \alpha_3\} \times$$
$$\exp. i\pi \{2(A\alpha_2 + H\alpha_3) z_2 + 2(H\alpha_2 + B\alpha_3) z_3 + (A, H, B)(\alpha_2, \alpha_3)^2\}.$$

The second line is here

$$= (-)^{\mu c_0 + \nu c_1 + q c_2 + r c_3} \exp. i\pi \{-2c_2 X - 2c_3 Y - (A, H, B)(c_2, c_3)^2\},$$

and the whole expression should be

$$= \exp. i\pi \{-2kx - kA'\}.$$

Hence bringing these terms over to the left-hand side, the equation will be satisfied if only the whole argument of the exponential be $= 0$; viz. omitting the factor $i\pi$, the equation will be satisfied if only

$$\begin{aligned}\alpha_0 z_2 + \alpha_2 z_0 + \alpha_1 z_3 + \alpha_3 z_1 + \alpha_0 \alpha_2 + \alpha_1 \alpha_3 - c_0 c_2 - c_1 c_3 & \\ + 2(A, H, B)(\alpha_2, \alpha_3)(z_2, z_3) + (A, H, B)(\alpha_2, \alpha_3)^2 & \\ - 2c_2 X - 2c_3 Y \qquad\qquad - (A, H, B)(c_2, c_3)^2 + k(2x + A') &= 0.\end{aligned}$$

Substituting here for X, Y their values

$$z_0 + Az_2 + Hz_3, \quad z_1 + Hz_2 + Bz_3,$$

and attending to the values

$$a_0 x + b_0 y, \quad a_1 x + b_1 y, \quad a_2 x + b_2 y, \quad a_3 x + b_3 y$$

of z_0, z_1, z_2, z_3, the equation contains a term in x, a term in y, and a constant term; and we may consider these terms separately.

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50. The term in x will vanish if

$$\alpha_0 a_2 + \alpha_2 a_0 + \alpha_3 a_1 + \alpha_1 a_3 + 2(A\alpha_2 + H\alpha_3) a_2 + 2(H\alpha_2 + B\alpha_3) a_3$$
$$- 2c_2 (a_0 + Aa_2 + Ha_3) - 2c_3 (a_1 + Ha_2 + Ba_3) + 2k = 0,$$

and substituting herein for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ their values

$$A'a_0 + H'b_0, \quad A'a_1 + H'b_1, \quad A'a_2 + H'b_2, \quad A'a_3 + H'b_3,$$

the equation becomes

$$A'[2a_0a_2 + 2a_1a_3 + 2(Aa_2 + Ha_3) a_2 + 2(Ha_2 + Ba_3) a_3]$$
$$+ H'[a_0b_2 + a_2b_0 + a_1b_3 + a_3b_1 + 2(Aa_2 + Ha_3) b_2 + 2(Ha_2 + Ba_3) b_3]$$
$$- 2c_2 (a_0 + Aa_2 + Ha_3) - 2c_3 (a_1 + Ha_2 + Ba_3) + 2k = 0,$$

or observing that the first and second lines may be expressed in the form

$$A'[2a_2 (a_0 + Aa_2 + Ha_3) + 2a_3 (a_1 + Ha_2 + Ba_3)]$$
$$+ H'[2b_2 (a_0 + Aa_2 + Ha_3) + 2b_3 (a_1 + Ha_2 + Ba_3)] - (a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1),$$

where the term $a_0b_2 - a_2b_0 + a_1b_3 - a_3b_1$, that is, $(02 + 13)_{ab}$, is $= 0$, and may therefore be omitted, the whole equation, omitting the factor 2, which divides out, is

$$(A'a_2 + H'b_2 - c_2)(a_0 + Aa_2 + Ha_3) + (A'a_3 + H'b_3 - c_3)(a_1 + Ha_2 + Ba_3) + k = 0,$$

an equation which, in the form

$$(-c_2T' + a_2P' + b_2Q')(a_0T + a_2P + a_3Q) + (-c_3T' + a_3P' + b_3Q')(a_1T + a_2Q + a_3R) + kTT' = 0,$$

has been above shown to be true. The term in x thus vanishes.

51. The term in y will vanish if

$$\alpha_0 b_2 + \alpha_2 b_0 + \alpha_1 b_3 + \alpha_3 b_1 + 2(A\alpha_2 + H\alpha_3) b_2 + 2(H\alpha_2 + B\alpha_3) b_3$$
$$- 2c_2 (b_0 + Ab_2 + Hb_3) - 2c_3 (b_1 + Hb_2 + Bb_3) = 0;$$

and this is in a similar manner reduced to

$$(A'a_2 + H'b_2 - c_2)(b_0 + Ab_2 + Hb_3) + (A'a_3 + H'b_3 - c_3)(b_1 + Hb_2 + Bb_3) = 0,$$

an equation which, in the form

$$(-c_2T' + a_2P' + b_2Q')(b_0T + b_2P + b_3Q) + (-c_3T' + a_3P' + b_3Q')(b_1T + b_2Q + b_3R) = 0,$$

has been above shown to be true. The term in y thus vanishes.

52. It only remains to show that the constant term also vanishes, viz. that we have

$$\alpha_0\alpha_2 + \alpha_1\alpha_3 + (A, H, B)(\alpha_2, \alpha_3)^2$$
$$- c_0c_2 - c_1c_3 - (A, H, B)(c_2, c_3)^2 + kA' = 0.$$

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Substituting here for $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, their values

$$= A'a_0 + H'b_0, \quad A'a_1 + H'b_1, \quad A'a_2 + H'b_2, \quad A'a_3 + H'b_3,$$

the equation becomes

$$\begin{aligned} &A'^2 (a_0 a_2 + a_1 a_3) + A'H' (a_0 b_2 + a_2 b_0 + a_1 b_3 + a_3 b_1) + H'^2 (b_0 b_2 + b_1 b_3) \\ &\quad + A\,[A'^2 a_2^2 + 2A'H' a_2 b_2 + H'^2 b_2^2] \\ &\quad + 2H\,[A'^2 a_2 a_3 + A'H' (a_2 b_3 + a_3 b_2) + H'^2 b_2 b_3] \\ &\quad + B\,[A'^2 a_3^2 + 2A'H' a_3 b_3 + H'^2 b_3^2] \\ &\quad - c_0 c_2 - c_1 c_3 - (A, H, B \between c_2, c_3)^2 + kA' = 0. \end{aligned}$$

This may be written

$$\begin{aligned} &A'^2 \{a_2 (a_0 + Aa_2 + Ha_3) + a_3 (a_1 + Ha_2 + Ba_3)\} \\ &+ A'H' \{b_2 (a_0 + Aa_2 + Ha_3) + b_3 (a_1 + Ha_2 + Ba_3) \\ &\qquad + a_2 (b_0 + Ab_2 + Hb_3) + a_3 (b_1 + Hb_2 + Bb_3)\} \\ &+ H'^2 \{b_2 (b_0 + Ab_2 + Hb_3) + b_3 (b_1 + Hb_2 + Bb_3)\} \\ &\qquad - c_2 (c_0 + Ac_2 + Hc_3) - c_3 (c_1 + Hc_2 + Bc_3) + kA' = 0. \end{aligned}$$

Writing herein

$$\begin{aligned} c_0 + Ac_2 + Hc_3 &= (a_0 + Aa_2 + Ha_3) A' + (b_0 + Ab_2 + Hb_3) H', \\ c_1 + Hc_2 + Bc_3 &= (a_1 + Ha_2 + Ba_3) A' + (b_1 + Hb_2 + Bb_3) H', \end{aligned}$$

equations which are true in virtue of

$$\begin{aligned} -(c_0 T + c_2 P + c_3 Q) T' + (a_0 T + a_2 P + a_3 Q) P' + (b_0 T + b_2 P + b_3 Q) Q' &= 0, \\ -(c_1 T + c_2 Q + c_3 R) T' + (a_1 T + a_2 Q + a_3 R) P' + (b_1 T + b_2 Q + b_3 R) Q' &= 0, \end{aligned}$$

the equation becomes

$$\begin{aligned} &(a_0 + Aa_2 + Ha_3) A' (A'a_2 + H'b_2 - c_2) \\ &+ (a_1 + Ha_2 + Ba_3) A' (A'a_3 + H'b_3 - c_3) \\ &+ (b_0 + Ab_2 + Hb_3) H' (A'a_2 + H'b_2 - c_2) \\ &+ (b_1 + Hb_2 + Bb_3) H' (A'a_3 + H'b_3 - c_3) + kA' = 0, \end{aligned}$$

that is,

$$\begin{aligned} &A' [(a_0 + Aa_2 + Ha_3)(-c_2 + A'a_2 + H'b_2) + (a_1 + Ha_2 + Ba_3)(-c_3 + A'a_3 + H'b_3) + k] \\ &+ H' [(b_0 + Ab_2 + Hb_3)(-c_2 + A'a_2 + H'b_2) + (b_1 + Hb_2 + Bb_3)(-c_3 + A'a_3 + H'b_3)] = 0; \end{aligned}$$

and we have the coefficients of A' and H' each $= 0$, in virtue of

$$\begin{aligned} (a_0 T + a_2 P + a_3 Q)(-c_2 T' + a_2 P' + b_2 Q') + (a_1 T + a_2 Q + a_3 R)(-c_3 T' + a_3 P' + b_3 Q') + kTT' &= 0, \\ (b_0 T + b_2 P + b_3 Q)(-c_2 T' + a_2 P' + b_2 Q') + (b_1 T + b_2 Q + b_3 R)(-c_3 T' + a_3 P' + b_3 Q') &= 0. \end{aligned}$$

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53. This completes the proof of the equation

$$\Pi(x+A',\ y+H') = (-)^{q'}\,\Pi(x,\ y)\ \text{exp.} -i\pi k(2x+A');$$

the proof of the remaining equation

$$\Pi(x+H',\ y+B') = (-)^{r'}\,\Pi(x,\ y)\ \text{exp.} -i\pi k(2y+B'),$$

is of course precisely similar.

It has already been seen that $\mu'q'+\nu'r' = \mu q+\nu r$; we have

$$\Theta(-X,\ -Y) = (-)^{\mu q+\nu r}\,\Theta(X,\ Y),$$

and thence

$$\Pi(-x,\ -y) = (-)^{\mu q+\nu r}\,\Pi(x,\ y),$$

that is,

$$\Pi(-x,\ -y) = (-)^{\mu' q'+\nu' r'}\,\Pi(x,\ y),$$

which is the last of the equations which should be satisfied by the function $\Pi(x,\ y)$.

Recapitulation, and Final Form. Art. Nos. 54 to 58.

54. Recapitulating, we have a Hermitian k-matrix (k an odd prime)

$$\begin{array}{|cccc|} \multicolumn{4}{c}{(\ a_0,\ \ a_1,\ \ a_2,\ \ a_3\ ),} \\ b_0, & b_1, & b_2, & b_3 \\ c_0, & c_1, & c_2, & c_3 \\ d_0, & d_1, & d_2, & d_3 \end{array}$$

susceptible of $(1+k+k^2+k^3)$ forms; further, writing for shortness

$$\begin{aligned} z_0 &= a_0x + b_0y, \\ z_1 &= a_1x + b_1y, \\ z_2 &= a_2x + b_2y, \\ z_3 &= a_3x + b_3y, \end{aligned}$$

and then

$$\begin{aligned} X &= z_0 + Az_2 + Hz_3, \\ Y &= z_1 + Hz_2 + Bz_3, \end{aligned}$$

and assuming

$$\Pi(x,\ y) = \Theta\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(X,\ Y)(A,\ H,\ B)\ \text{exp.}\ i\pi\{(z_0z_2+z_1z_3)+(A,\ H,\ B)(z_2,\ z_3)^2\},$$

then $\Pi(x,\ y)$ satisfies the equations

$$\begin{aligned} \Pi(x+1,\ \ y\ \ ) &= (-)^{\mu'}\,\Pi(x,\ y), \\ \Pi(\ \ x\ \ ,\ y+1) &= (-)^{\nu'}\,\Pi(x,\ y), \\ \Pi(x+A',\ y+H') &= (-)^{q'}\,\Pi(x,\ y)\ \text{exp.} -i\pi k(2x+A'), \\ \Pi(x+H',\ y+B') &= (-)^{r'}\,\Pi(x,\ y)\ \text{exp.} -i\pi k(2y+B'), \\ \Pi(-x,\ -y) &= (-)^{\mu'q'+\nu'r'}\,\Pi(x,\ y), \end{aligned}$$

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and it is consequently a function

$$\Pi\begin{pmatrix}\mu', & \nu' \\ q', & r'\end{pmatrix}(x,\ y)(A',\ H',\ B')\,\{K,\ \text{def.}\},$$

and as such, contains linearly $\frac{1}{2}(k^2+1)$ constants.

55. The values of the new parameters are given as above, viz. T', P', Q', R', S' being linear functions of T, P, Q, R, S (the coefficients in these relations being given functions of the coefficients $(a,\ b,\ c,\ d)_{0,\,1,\,2,\,3}$ of the Hermitian matrix), then

$$1 : A : H : B : H^2 - AB = T : P : Q : R : S,$$

$$1 : A' : H' : B' : H'^2 - A'B' = T' : P' : Q' : R' : S',$$

which represent the required relations.

56. We consider four such functions $\Pi(x,\ y)$, derived from Θ-functions having respectively the characteristics

$$\begin{pmatrix}\mu_0, & \nu_0 \\ q_0, & r_0\end{pmatrix},\quad \begin{pmatrix}\mu_1, & \nu_1 \\ q_1, & r_1\end{pmatrix},\quad \begin{pmatrix}\mu_2, & \nu_2 \\ q_2, & r_2\end{pmatrix},\quad \begin{pmatrix}\mu_3, & \nu_3 \\ q_3, & r_3\end{pmatrix},$$

being a (0123) system; and having consequently the characteristics

$$\begin{pmatrix}\mu_0', & \nu_0' \\ q_0', & r_0'\end{pmatrix},\quad \begin{pmatrix}\mu_1', & \nu_1' \\ q_1', & r_1'\end{pmatrix},\quad \begin{pmatrix}\mu_2', & \nu_2' \\ q_2', & r_2'\end{pmatrix},\quad \begin{pmatrix}\mu_3', & \nu_3' \\ q_3', & r_3'\end{pmatrix},$$

which also form a (0123) system; say the four are Π_0, Π_1, Π_2, Π_3.

This being so, consider four theta-functions

$$\theta(x,\ y)(A',\ H',\ B'),$$

having respectively the characteristics

$$\begin{pmatrix}\mu_0', & \nu_0' \\ q_0', & r_0'\end{pmatrix},\quad \begin{pmatrix}\mu_1', & \nu_1' \\ q_1', & r_1'\end{pmatrix},\quad \begin{pmatrix}\mu_2', & \nu_2' \\ q_2', & r_2'\end{pmatrix},\quad \begin{pmatrix}\mu_3', & \nu_3' \\ q_3', & r_3'\end{pmatrix},$$

which as already mentioned form a (0123) system; form with these the four sums

$$\Sigma\theta_0^{\mathrm{a}}\theta_1^{\mathrm{b}}\theta_2^{\mathrm{c}}\theta_3^{\mathrm{d}},$$

where in each case $\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}=k$, but in the first sum a, in the second sum b, in the third sum c, and in the fourth sum d is of contrary parity to the other three letters (even, if they are odd; odd, if they are even), say the four sums are Σ_0, Σ_1, Σ_2, Σ_3 respectively; each sum depends linearly on $\frac{1}{2}(k^2+1)$ constants.

The general transformation theorem then is

$$\Sigma_0 = \Pi_0,\quad \Sigma_1 = \Pi_1,\quad \Sigma_2 = \Pi_2,\quad \Sigma_3 = \Pi_3;$$

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each of which equations in fact represents $\frac{1}{2}(k^2+1)$ equations; viz. on the left-hand side we assign to the constants $\frac{1}{2}(k^2+1)$ systems of values at pleasure, then to each such system there corresponds on the right-hand side a determinate system of values of the $\frac{1}{2}(k^2+1)$ constants.

57. The results may be presented in a more symmetrical form. Writing ξ, η, ζ, ω for the foregoing x, y, z, w, we may consider

$$\Theta\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(\xi + A'\zeta + H'\omega,\ \eta + H'\zeta + B'\omega)(A',\ H',\ B')$$

as a function of the four arguments ξ, η, ζ, ω, and express it by

$$\Phi\begin{pmatrix}\mu, & \nu\\ q, & r\end{pmatrix}(\xi,\ \eta,\ \zeta,\ \omega)(A',\ H',\ B').$$

Writing then

$$x = \xi + A'\zeta + H'\omega,$$

$$y = \eta + H'\zeta + B'\omega,$$

$$X,\ Y,\ Z,\ W = \left(\begin{array}{cccc} a_0, & b_0, & c_0, & d_0 \\ a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{array}\right)(\xi,\ \eta,\ \zeta,\ \omega),$$

we have

$$\begin{aligned} z_1 + Az_2 + Hz_3 &= a_0x + b_0y + A\,(a_2x + b_2y) + H\,(a_3x + b_3y) \\ &= (a_0 + Aa_2 + Ha_3)(\xi + A'\zeta + H'\omega) \\ &\quad + (b_0 + Bb_2 + Hb_3)(\eta + H'\zeta + B'\omega), \end{aligned}$$

and also $X + AZ + HW =$

$$\begin{array}{lll} a_0\xi + b_0\eta + c_0\zeta + d_0\omega & = & (a_0 + Aa_2 + Ha_3)\,\xi \\ +A\,(a_2\xi + b_2\eta + c_2\zeta + d_2\omega) & & +(b_0 + Ab_2 + Hb_3)\,\eta \\ +H\,(a_3\xi + b_3\eta + c_3\zeta + d_3\omega) & & +(c_0 + Ac_2 + Hc_3)\,\zeta \\ & & +(d_0 + Ad_2 + Hd_3)\,\omega, \end{array}$$

or since, as above,

$$A'(a_0 + Aa_2 + Ha_3) + H'(b_0 + Ab_2 + Hb_3) = c_0 + Ac_2 + Hc_3,$$

$$H'(a_0 + Aa_2 + Ha_3) + B'(b_0 + Ab_2 + Hb_3) = d_0 + Ad_2 + Hd_3,$$

we find

$$z_0 + Az_2 + Hz_3 = X + AZ + HW;$$

and in like manner another equation, viz. we have

$$z_0 + Az_2 + Hz_3 = X + AZ + HW,$$

$$z_1 + Hz_2 + Bz_3 = Y + HZ + BW.$$

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Moreover

$$\begin{aligned}
&z_0z_2 + z_1z_3 + (A,\ H,\ B\between z_2,\ z_3)^2 \\
&\quad = \quad z_2(z_0 + Az_2 + Hz_3) = \qquad (a_2x + b_2y)(X + AZ + HW) \\
&\qquad + z_3(z_1 + Hz_2 + Bz_3) \qquad + (a_3x + b_3y)(Y + HZ + BW) \\
&\qquad\qquad\qquad = \quad a_2(\xi + A'\zeta + H'\omega)(X + AZ + HW) \\
&\qquad\qquad\qquad\quad + b_2(\eta + H'\zeta + B'\omega)(\quad ,, \quad) \\
&\qquad\qquad\qquad\quad + a_3(\xi + A'\zeta + H'\omega)(Y + HZ + BW) \\
&\qquad\qquad\qquad\quad + b_3(\eta + H'\zeta + B'\omega)(\quad ,, \quad),
\end{aligned}$$

which *quà* function of X, Y, Z, W may be called χ.

58. And we then have the final result in the following form, viz.

$$\text{exp. } i\pi\chi \,.\, \Phi\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(X,\ Y,\ Z,\ W)(A,\ H,\ B),$$

in each of the four forms, is a homogeneous function of the order k of the corresponding four forms

$$\Phi\begin{pmatrix}\mu, & \nu \\ q, & r\end{pmatrix}(\xi,\ \eta,\ \zeta,\ \omega)(A',\ H',\ B').$$

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