842.

ON THE VALUE OF $\tan(\sin\theta) - \sin(\tan\theta)$.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 191, 192.]

THE following equation is given p. 59 of the *Lady's and Gentleman's Diary* for 1853:

$$\tan(\sin\theta) - \sin\tan\theta = \tfrac{1}{30}\theta^7 + \tfrac{29}{756}\theta^9 + \&c.$$

Write in general

$$X = \theta + A\theta^3 + B\theta^5 + C\theta^7 + D\theta^9 + \ldots,$$
$$Y = \theta + A'\theta^3 + B'\theta^5 + C'\theta^7 + D'\theta^9 + \ldots.$$

Then, as far as θ^9, we have

$$\begin{aligned} & X + A'X^3 + B'X^5 + C'X^7 + D'X^9 \\ = {} & \theta + A\theta^3 + B\theta^5 + C\theta^7 + D\theta^9 \\ & + A'\{\theta^3 + 3A\theta^5 + 3(A^2+B)\theta^7 + (A^3 + 6AB + 3C)\theta^9\} \\ & + B'\{\theta^5 + 5A\theta^7 + (10A^2 + 5B)\theta^9\} \\ & + C'\{\theta^7 + 7A\theta^9\} \\ & + D'\{\theta^9\} \\ = {} & +\theta \\ & + \theta^3(A + A') \\ & + \theta^5(B + 3AA' + B') \\ & + \theta^7(C + 3A^2A' + 3A'B + 5AB' + C') \\ & + \theta^9(D + A^3A' + 6AA'B + 3A'C + 10A^2B' + 5BB' + 7AC + D'), \end{aligned}$$

and hence

$$X + A'X^3 + B'X^5 + C'X^7 + D'X^9$$
$$- Y - AY^3 - BY^5 - CY^7 - DY^9$$
$$= \theta^7 \{3AA'(A - A') + 2(AB' - A'B)\}$$
$$+ \theta^9 \{AA'(A^2 - A'^2) + 6AA'(B - B') + 4(AC' - A'C) + 10(A^2B' - A'^2B)\}.$$

Now let

$$X = \sin\theta = \tfrac{1}{6}\theta^3 + \tfrac{1}{120}\theta^5 - \tfrac{1}{5040}\theta^7 + \dots; \quad A = -\tfrac{1}{6},\ B = \tfrac{1}{120},\ C = -\tfrac{1}{5040},$$
$$Y = \tan\theta = \tfrac{1}{3}\theta^3 + \tfrac{2}{15}\theta^5 + \tfrac{17}{315}\theta^7 - \dots; \quad A' = \tfrac{1}{3},\ B' = \tfrac{2}{15},\ C' = \tfrac{17}{315};$$

we have therefore

$$AA' = -\tfrac{1}{18},\quad A - A' = -\tfrac{1}{2},\quad A + A' = \tfrac{1}{6},\quad B - B' = -\tfrac{1}{8},$$
$$AB' - A'B = -\tfrac{1}{40},\quad AC' - A'C = -\tfrac{1}{112},\quad A^2B' - A'^2B = \tfrac{1}{360}.$$

Hence

$$\text{coeff. } \theta^7 = \tfrac{1}{12} - \tfrac{1}{20},\ = \tfrac{1}{30},$$
$$\text{coeff. } \theta^9 = \tfrac{1}{216} + \tfrac{1}{24} - \tfrac{1}{28} + \tfrac{1}{36},\ = \tfrac{29}{756};$$

and the required equation is thus verified.