

840.

ON MASCHERONI'S GEOMETRY OF THE COMPASS.

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I HAVE not seen the original of Mascheroni's work, *La Geometria del Compasso*, 8°, Pavia (1797), but only the French translation, *Géométrie du Compas*, par L. Mascheroni, traduite de l'Italien par A. M. Carette, 2 ed. 8° Paris, 1828 (Author's Preface, pp. 5—24, and pp. 25—328, with 14 plates containing 108 figures). The title expresses the notion of the work: a straight line is given by means of two points thereof, but is not allowed to be actually drawn; and the problem is, with the compass alone to perform all the constructions of Geometry. Observe that, for the purpose of demonstration, any lines may be imagined to be drawn, and such lines are in fact shown as dotted lines in the figures; but this does not in any wise interfere with the fundamental postulate that the constructions are to be performed with the compass only.

Assuming, then, that a line is in every case given by means of two points thereof, the leading questions are those considered Book VII. "on the intersections of straight lines with circular arcs and with each other," viz. they are (1) to find the intersections of a given line and circle; (2) to find the intersection of two given lines. But in the first problem it is necessary to consider two cases; (*A*) the general case, where the line does not pass through the centre of the circle, and (*B*) the particular, but actually more difficult, case, where the line passes through the centre of the circle. It is assumed that the centre of the circle is given: if it is not given, it can be found as afterwards mentioned.

It will be convenient to establish the definition of counter-points in regard to a given line: two points, which are such that the line joining them is bisected at right angles by the given line, are said to be counter-points in regard to that line.

(1), *A*. Consider a line given by means of two points P, Q ; and a given circle, centre C . With centre P and radius PC describe a circle, and with centre Q and

radius QC a circle; these meet in C and in a second point D which will be the counter-point of C in regard to the line PQ ; hence with centre D and radius = that of given circle, describing a circle, this meets the given circle in two points which are the intersections of the given circle by the line PQ .

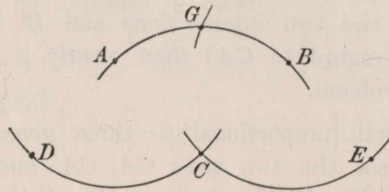
The construction fails if the line PQ passes through the centre of the given circle, for then the two auxiliary circles touch at C , or we have D coincident with C . We have therefore considered this case; Q may be taken to be the centre C of the circle; so that we have:—

(1), *B*. The problem is, given a point P , and a circle, centre C : to find the intersections of the circle by the line CP . With centre P , and an arbitrary radius, describe a circle meeting the given circle in points A, B (which are of course counter-points in regard to line CP); the circle is divided into two arcs AB and $360^\circ - AB$, and the problem is to bisect each of these two arcs; for the points of bisection are obviously the required intersections of CP with the circle.

Hence we have:—

(1), *C*. To bisect a given arc AB of a circle, centre C (fig. 1). With centre A and radius AC describe a circle; and with centre B , and equal radius BC describe a circle; and on these circles respectively, find the points D, E such that $CD = CE = AB$.

Fig. 1.



With centre E and radius EA , describe a circle; and with centre D , and equal radius DB , describe a circle; and let these two circles meet in F ; then with centre E (or D) and radius CF describe a circle: this will meet the given circle in the required middle point G of the arc AB .

The proof depends on the theorem that in a rhombus the sum of the squares of the diagonals is = sum of the squares of the four side. We have a rhombus $ACEB$, and therefore

$$(AE)^2 + (BC)^2 = 2(AB)^2 + 2(AC)^2,$$

that is,

$$(AE)^2 = 2(AB)^2 + \text{rad.}^2.$$

But by construction, $EF^2 = (AE)^2$; therefore

$$EF^2 = 2(AB)^2 + \text{rad.}^2, = 2(CE)^2 + \text{rad.}^2.$$

Hence in the right-angled triangle FCE , we have

$$(CF)^2 = (EF)^2 - CE^2, = (CE)^2 + \text{rad.}^2;$$

and by construction, $(EG)^2 = (CF)^2$; that is,

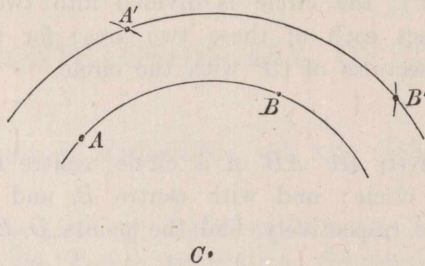
$$(EG)^2 = (CE)^2 + \text{rad.}^2 = (CE)^2 + (CG)^2;$$

viz. CEG is a right-angled triangle; that is, CG is at right angles to BCE , or the line CGF bisects the arc AB in G .

For the second problem, (2), to find the intersection of two given lines, we require the solution of the problem to find a fourth proportional to three given distances, and this immediately depends on the following problem.

Problem. In two concentric circles to place chords subtending equal angles. If AB (fig. 2) be a chord of the one circle, then with centres A and B respectively, and

Fig. 2.



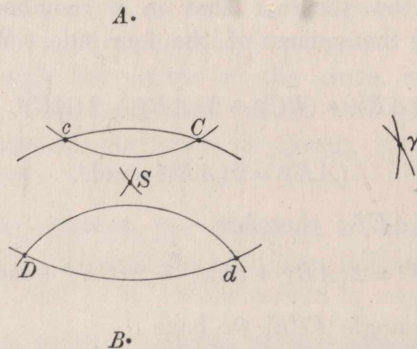
an arbitrary radius $AA' = BB'$, describing circles to cut the larger circle in the points A' and B' (A' is either of the two intersections and B' is the intersection lying in regard to CB as A' lies in regard to CA) then clearly $\angle ACB = \angle A'CB'$. And hence also we have the following problem.

Problem. To find a fourth proportional to three given distances. We have only to take as the given distances the two radii CA, CA' and the chord AB ; and then from the similar triangles $ACB, A'CB'$, we have $CA : CA' :: AB : A'B'$, viz. $A'B'$ is a fourth proportional to CA, CA', AB .

We have now the solution of

(2) To construct the intersection of two given lines AB and CD (fig. 3). Find

Fig. 3.



c the counter-point of C , and d the counter-point of D , in regard to the line AB :

and find γ such that the distances $C\gamma$, $d\gamma$ are $= Dd$, DC respectively; γ is in a line with cC , and we have $C\gamma dD$ a parallelogram. Find cS a fourth proportional to $c\gamma$, cd , cC , and with centres c , C respectively and radii each $= cS$ describe circles cutting in the point S ; this will be the required intersection of the two lines. In fact, the required point S will be the intersection of the two lines CD , cd : supposing these lines each of them drawn, and also the lines $cC\gamma$ and $d\gamma$, we have DC parallel to $d\gamma$, that is, the triangles $cd\gamma$, cSC are similar to each other or $c\gamma : cd :: cC : cS$: viz. the distance cS having been found by this proportion, and the point S found as the intersection of the two circles, centres c and C respectively, the point S so determined is the required point of intersection of the given lines AB and CD .

If a circle be given without its centre being known, then taking any three points A , B , C on the circle, and a pair of counter-points D , E of the line AB , and also a pair of counter-points F , G of the line AC , we have obviously the centre of the given circle as the intersection of the lines DE and FG ; and the centre can thus be found with the compass only.

It is proper to remark that the problems considered in the present paper are those connecting the theory with ordinary geometry, not the problems which are most readily and elegantly solved with the compass only: a large collection of these are contained in the work, and in particular the twelfth book contains some interesting *approximate* solutions of the problems of the quadrature of the circle, the duplication of the cube, and other problems not solvable by ordinary geometry.

Cambridge, March 19, 1885.