839.

ON THE MATRICAL EQUATION $qQ - Qq' = 0$.

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I CONSIDER the matrical equation $qQ - Qq' = 0$, where q, q' are given matrices $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$, $\begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix}$ and Q, $= \begin{vmatrix} A, & B \\ C, & D \end{vmatrix}$ is a matrix which has to be determined. As remarked in the paper "On the Quaternion Equation $qQ - Qq' = 0$," *Messenger*, t. XIV. (1885), pp. 108—112, [836] the question for matrices is equivalent to that for quaternions: for a matrix $\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$ may be regarded as a quaternion

$$\tfrac{1}{2}a\,(1 - \lambda i) + \tfrac{1}{2}b\,(j + \lambda k) + \tfrac{1}{2}c\,(-j - \lambda k) + \tfrac{1}{2}d\,(1 + \lambda i),$$

or (omitting the factor $\tfrac{1}{2}$) as the quaternion

$$(a + d) - \lambda\,(a - d)\,i + (b - c)\,j - \lambda\,(b + c)\,k,$$

where λ, $= \sqrt{(-1)}$, is the imaginary of ordinary algebra. Hence considering q, q' as denoting the quaternions

$$(a + d) - \lambda\,(a - d)\,i + (b - c)\,j - \lambda\,(b + c)\,k,$$

$$(a' + d') - \lambda\,(a' - d')\,i + (b' - c')\,j - \lambda\,(b' + c')\,k,$$

we can, if a certain condition is satisfied, find a quaternion Q such that $qQ - Qq' = 0$; say this is $Q = W + iX + jY + kZ$; putting this

$$= \tfrac{1}{2}\,\{(A + D) - \lambda(A - D)\,i + (B - C)\,j - \lambda\,(B + C)\,k\},$$

we find

$$Q = \begin{vmatrix} A, & B \\ C, & D \end{vmatrix}, \quad = \begin{vmatrix} W + \lambda X, & Y + \lambda Z \\ -Y + \lambda Z, & W - \lambda X \end{vmatrix},$$

for the required matrix Q; this being an indeterminate matrix, such that $AD - BC = 0$.

But it is better to solve directly the matrical equation

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} \begin{vmatrix} A, & B \\ C, & D \end{vmatrix} - \begin{vmatrix} A, & B \\ C, & D \end{vmatrix} \begin{vmatrix} a', & b' \\ c', & d' \end{vmatrix} = 0,$$

viz.

$$\begin{array}{c|cc} & (A,\ C), & (B,\ D) \\ \hline (a,\ b) & ,, & ,, \\ (c,\ d) & ,, & ,, \end{array} - \begin{array}{c|cc} & (a',\ c'), & (b',\ d') \\ \hline (A,\ B) & ,, & ,, \\ (C,\ D) & ,, & ,, \end{array} = 0,$$

that is,

$$\begin{aligned} Aa + Cb - (Aa' + Bc') &= 0, \\ Ba + Db - (Ab' + Bd') &= 0, \\ Ac + Cd - (Ca' + Dc') &= 0, \\ Bc + Dd - (Cb' + Dd') &= 0, \end{aligned}$$

or, what is the same thing,

$$\left(\begin{array}{cccc} a-a', & -c', & b\ , & 0\ , \\ -b', & a-d', & 0\ , & b\ , \\ c, & 0, & d-a', & -c', \\ 0, & c, & -b', & d-d', \end{array}\right)(A,\ B,\ C,\ D)=0,$$

viz. we have (A, B, C, D) connected by these four linear equations: viz. the necessary condition is that the determinant formed out of the matrix which here presents itself shall be $=0$.

After some reductions, and putting for shortness

$$\nabla = ad - bc, \quad \nabla' = a'd' - b'c',$$

this is found to be

$$(\nabla - \nabla')^2 + \{\nabla (a' + d') - \nabla' (a + d)\} (a' + d' - a - d) = 0;$$

which is the condition for the existence of a solution. This condition being satisfied, the four equations will be equivalent to three independent equations, which serve to determine A, B, C, D; and, assuming the absolute value of A, we find

$$\begin{aligned} A &= \qquad\qquad - (a' + d' - a - d), \\ c'B &= \nabla - \nabla' + a' (a' + d' - a - d), \\ bC &= \nabla - \nabla' + a\ (a' + d' - a - d), \\ bc'D &= (d' - a)\ \nabla - (d - a')\ \nabla' - aa' (a' + d' - a - d), \end{aligned}$$

values which give

$$-bc' (AD - BC) = (\nabla - \nabla')^2 + \{\nabla (a' + d') - \nabla' (a + d)\} (a' + d' - a - d),\ = 0,$$

viz. in the case where the given matrices satisfy the above-mentioned condition, the components A, B, C, D of the required matrix have determinate values which are such that $AD - BC = 0$.

If we have $\nabla - \nabla' = 0$, that is, $ad - bc = a'd' - b'c'$, then the condition becomes $a' + d' = a + d$; and these two conditions being satisfied, the four equations reduce themselves to two independent equations, and the ratios $A : B : C : D$ have no longer determinate values; the linear equations may in fact be written

$$\left(\begin{array}{cccc} 0, & b, & -c', & a-a' \\ -b, & 0, & -a+d', & b' \\ c', & a'-d, & 0, & -c \\ d-d', & -b', & c, & 0 \end{array}\right)(D, C, B, A) = 0,$$

which are of the form

$$\left(\begin{array}{cccc} 0, & \mathrm{h}, & -\mathrm{g}, & \mathrm{a} \\ -\mathrm{h}, & 0, & \mathrm{f}, & \mathrm{b} \\ \mathrm{g}, & -\mathrm{f}, & 0, & \mathrm{c} \\ -\mathrm{a}, & -\mathrm{b}, & -\mathrm{c}, & 0 \end{array}\right)(D, C, B, A) = 0,$$

where the coefficients a, b, c, f, g, h are such that $\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0$; and the four equations are thus equivalent to two independent equations. To obtain a symmetrical solution, assume a relation

$$D\xi + C\eta + B\zeta + A\omega = 0,$$

with arbitrary coefficients ξ, η, ζ, ω; we then find

$$\begin{aligned} D &= \quad . \quad \mathrm{c}\eta - \mathrm{b}\zeta + \mathrm{f}\omega, \\ C &= -\mathrm{c}\xi \quad . \quad + \mathrm{a}\zeta + \mathrm{g}\omega, \\ B &= \mathrm{b}\xi - \mathrm{a}\eta \quad . \quad + \mathrm{h}\omega, \\ A &= -\mathrm{f}\xi - \mathrm{g}\eta - \mathrm{h}\zeta \quad . \;, \end{aligned}$$

viz. the complete theorem is that, if the matrices

$$q = \left|\begin{array}{cc} a, & b \\ c, & d \end{array}\right|, \quad q' = \left|\begin{array}{cc} a', & b' \\ c', & d' \end{array}\right|$$

are such that $ad - bc = a'd' - b'c'$, $a + d = a' + d'$, then writing

$$\begin{aligned} \mathrm{a} &= a - a', = -d + d', \quad & \mathrm{f} &= -a + d', = -a' + d, \\ \mathrm{b} &= b', & \mathrm{g} &= -c', \\ \mathrm{c} &= -c, & \mathrm{h} &= b, \end{aligned}$$

values which satisfy

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0,$$

and taking ξ, η, ζ, ω arbitrary, we have for the matrix Q, which is such that $Qq - Qq' = 0$, the expression

$$Q = \left|\begin{array}{cc} -\mathrm{f}\xi - \mathrm{g}\eta - \mathrm{h}\zeta, & \mathrm{b}\xi - \mathrm{a}\eta \;\; . \; + \mathrm{h}\omega \\ -\mathrm{c}\xi \;\; . \; + \mathrm{a}\zeta + \mathrm{g}\omega, & \mathrm{c}\eta - \mathrm{b}\zeta + \mathrm{f}\omega \end{array}\right|,$$

depending really on one arbitrary parameter, viz. we may without loss of generality put any two of the coefficients ξ, η, ζ, ω equal to 0.