835.

ON CARDAN'S SOLUTION OF A CUBIC EQUATION.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 96, 97.]

IT is interesting to see how the solution comes out when one root of the equation is known. Say the equation is $x^3 + qx - r = 0$, where $a^3 - qa - r = 0$, that is, $r = a^3 + qa$.

Solving in the usual manner, we have

$$x = y + z, \quad y^3 + z^3 - r + (y + z)(3yz + q) = 0,$$

whence

$$y^3 + z^3 = r,$$
$$yz = -\tfrac{1}{3}q;$$

and thence

$$(y^3 - z^3)^2 = r^2 + \tfrac{4}{27}q^3, \quad = a^6 + 2qa^4 + q^2a^2 - \tfrac{4}{27}q^3, \quad = (a^2 + \tfrac{4}{3}q)(a^2 + \tfrac{1}{3}q)^2;$$

or say

$$y^3 - z^3 = (a^2 + \tfrac{1}{3}q)\sqrt{(a^2 + \tfrac{4}{3}q)};$$

and therefore

$$8y^3 = 4a^3 + 4qa + (4a^2 + \tfrac{4}{3}q)\sqrt{(a^2 + \tfrac{4}{3}q)}, \quad = \{a + \sqrt{(a^2 + \tfrac{4}{3}q)}\}^3,$$
$$8z^3 = 4a^3 + 4qa - (4a^2 + \tfrac{4}{3}q)\sqrt{(a^2 + \tfrac{4}{3}q)}, \quad = \{a - \sqrt{(a^2 + \tfrac{4}{3}q)}\}^3;$$

where observe that the essential step is the expression of the irrational functions as perfect cubes: that the functions are the cubes of $a \pm \sqrt{(a^2 + \tfrac{4}{3}q)}$ respectively is seen to be true; but if we were to attempt to find a cube root $\alpha + \beta\sqrt{(a^2 + \tfrac{4}{3}q)}$ by an algebraical process, we should be thrown back upon the original cubic equation.

Writing then ω for an imaginary cube root of unity, we have

$$2y = (1,\ \omega \ \text{or}\ \omega^2)\{a + \sqrt{(a^2 + \tfrac{4}{3}q)}\},$$
$$2z = (1,\ \omega^2 \ \text{or}\ \omega)\{a - \sqrt{(a^2 + \tfrac{4}{3}q)}\};$$

and then

$$x = y + z = a, \ \text{or} \ = -\tfrac{1}{2}a \pm \tfrac{1}{2}(\omega - \omega^2)\sqrt{(a^2 + \tfrac{4}{3}q)},$$

where $\omega - \omega^2 = i\sqrt{3}$; the last two roots are of course the roots of the quadric equation $x^2 + ax + a^2 + q = 0$, which is obtained by throwing out the factor $x - a$ from the given equation $x^3 + qx - r = 0$.

Cambridge, Sep. 17, 1884.