

834.

ON THE ADDITION OF THE ELLIPTIC FUNCTIONS.

[From the *Messenger of Mathematics*, vol. XIV. (1885), pp. 56—61.]

MR FORSYTH'S Note [*l. c.*, p. 23] on my "Formula in Elliptic Functions" has supplied a missing link, and I am now able to obtain the addition formulæ very simply from the application of Abel's theorem to the Quadriquadric Curve.

I remark that, instead of coplanar points 1, 2, 3, 4, it is advantageous to consider coresidual points 1, 2 and 3, 4; that is, pairs 1, 2 and 3, 4, which are each of them coplanar with one and the same pair of points 5, 6. The difference is as follows: for the coplanar points 1, 2, 3, 4, we have

$$du_1 + du_2 + du_3 + du_4 = 0,$$

giving

$$u_1 + u_2 + u_3 + u_4 = C,$$

and for the addition theory it is necessary to have $C = 0$; for the coresidual points, we have

$$u_1 + u_2 + u_5 + u_6 = C, \quad u_3 + u_4 + u_5 + u_6 = C;$$

and thence $u_1 + u_2 = u_3 + u_4$, irrespectively of the value of C .

As to the general theory of a curve in space, observe that, when this is a complete intersection of two surfaces

$$f(x, y, z, w) = 0, \quad g(x, y, z, w) = 0,$$

then at the point (x, y, z, w) , if

$$(x + dx, y + dy, z + dz, w + dw)$$

are the coordinates of the consecutive point, the six coordinates of the tangent line are

$$y dz - z dy, \quad z dx - x dz, \quad x dy - y dx, \quad x dw - w dx, \quad y dw - w dy, \quad z dw - w dz.$$

But considering the line as the intersection of the two tangent planes

$$\frac{df}{dx} X + \frac{df}{dy} Y + \frac{df}{dz} Z + \frac{df}{dw} W = 0,$$

and

$$\frac{dg}{dx} X + \frac{dg}{dy} Y + \frac{dg}{dz} Z + \frac{dg}{dw} W = 0,$$

the six coordinates are

$$\frac{d(f, g)}{d(x, w)}, \quad \frac{d(f, g)}{d(y, w)}, \quad \frac{d(f, g)}{d(z, w)}, \quad \frac{d(f, g)}{d(y, z)}, \quad \frac{d(f, g)}{d(z, x)}, \quad \frac{d(f, g)}{d(x, y)},$$

so that the six quotients

$$(y dz - z dy) \left/ \frac{d(f, g)}{d(x, w)}, \right. \&c.,$$

are equal to each other, and may be put $= d\omega$.

Considering any two quadric surfaces, there is in general a system of four conjugate points, or points such that in regard to each of the quadrics the polar plane of any one of the points is the plane through the other three points. And then taking $x=0, y=0, z=0, w=0$ for the equations of the faces of the tetrahedron formed by the four points, the equations of the quadric surfaces will be of the form

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$

$$a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0;$$

we then have the six quotients

$$(y dz - z dy) / (ad' - a'd) xw, \&c.,$$

equal to each other, and each $= d\omega$. Here $d\omega$ is homogeneous of the degree zero in the coordinates (x, y, z, w) , or, what is the same thing, it is a differential $F\left(\frac{x}{w}\right) d\frac{x}{w}$, say it is $= du$; and taking the integrals always from one and the same fixed point on the curve, we have each point of the curve corresponding to a determinate value of a parameter u .

Supposing that u_1, u_2, u_5, u_6 are the values of u , belonging to any four coplanar points 1, 2, 5, 6; then, by Abel's theorem, $du_1 + du_2 + du_5 + du_6 = 0$; that is, we have

$$u_1 + u_2 + u_5 + u_6 = C,$$

as the condition in order that the four points 1, 2, 5, 6 may be coplanar; similarly, we have

$$u_3 + u_4 + u_5 + u_6 = C,$$

as the condition in order that the four points 3, 4, 5, 6 may be coplanar; and we have therefore

$$u_1 + u_2 = u_3 + u_4,$$

as the condition that the two pairs of points 1, 2 and 3, 4 may be coresidual.

The points 1, 2, 5, 6 are coplanar, hence the line 56 meets the line 12, say in the point *A*; and the points 3, 4, 5, 6 are coplanar, hence the line 56 meets the line 34, say in the point *B*. We can, through the curve and any arbitrary point in space, draw a quadric surface

$$(a + \lambda a')x^2 + (b + \lambda b')y^2 + (c + \lambda c')z^2 + (d + \lambda d')w^2 = 0.$$

Hence we have such a quadric surface through the point *A*; and this surface, passing through 5 and 6, will contain the line 56, and therefore also the point *B*; hence, passing through 3 and 4, it will contain the line 34; viz. we have the lines 12, 34 as generating lines, obviously of the same kind, on the last-mentioned quadric surface. I say that if, on such a surface, that is, on any surface

$$Ax^2 + By^2 + Cz^2 + Dw^2 = 0,$$

we have

$$(a, b, c, f, g, h), \quad (a', b', c', f', g', h'),$$

the coordinates of two generating lines of the same kind, then

$$\frac{a}{f} = \frac{a'}{f'}, \quad \frac{b}{g} = \frac{b'}{g'}, \quad \frac{c}{h} = \frac{c'}{h'}.$$

This is at once seen to be the case; for, taking θ an arbitrary parameter, we have for the equations of a generating line

$$\begin{aligned} & \{x\sqrt{A} + iy\sqrt{B}\} + \theta \{z\sqrt{C} + iw\sqrt{D}\} = 0, \\ & \theta \{x\sqrt{A} - iy\sqrt{B}\} - \{z\sqrt{C} - iw\sqrt{D}\} = 0, \end{aligned}$$

and the coordinates (a, b, c, f, g, h) of this line are

$$\begin{aligned} & i\sqrt{AD}(1 - \theta^2), \quad \sqrt{BD}(-1 - \theta^2), \quad i\sqrt{CD}2\theta, \\ & i\sqrt{BC}(-1 + \theta^2), \quad \sqrt{CA}(1 + \theta^2), \quad i\sqrt{AB}(-2\theta); \end{aligned}$$

that is, the quotients $\frac{a}{f}, \frac{b}{g}, \frac{c}{h}$ are each of them independent of θ ; and they have consequently their values unaltered when for the original line we substitute any other generating line of the same kind. Or, to prove the statement in a different manner, the equation of the quadric surface through the line (a, b, c, f, g, h) is

$$aghx^2 + bhfy^2 + cfgz^2 + abcw^2 = 0;$$

hence, if this contains the line (a', b', c', f', g', h') , we must have

$$agh : bhf : cfy : abc = a'g'h' : b'h'f' : c'f'g' : a'b'c',$$

equations which give either

$$af' + a'f = 0, \quad bg' + b'g = 0, \quad ch' + c'h = 0,$$

or else

$$af' - a'f = 0, \quad bg' - b'g = 0, \quad ch' - c'h = 0.$$

In the former case, the two lines are generating lines of different kinds; in the latter, they are generating lines of the same kind.

Now, considering (a, b, c, f, g, h) as the coordinates of the line 12 and (a', b', c', f', g', h') as those of the line 34, the equations just obtained are

$$\frac{y_1 z_2 - y_2 z_1}{x_1 w_2 - x_2 w_1} = \frac{y_3 z_4 - y_4 z_3}{x_3 w_4 - x_4 w_3}, \quad \frac{z_1 x_2 - z_2 x_1}{y_1 w_2 - y_2 w_1} = \frac{z_3 x_4 - z_4 x_3}{y_3 w_4 - y_4 w_3}, \quad \frac{x_1 y_2 - x_2 y_1}{z_1 w_2 - z_2 w_1} = \frac{x_3 y_4 - x_4 y_3}{z_3 w_4 - z_4 w_3}.$$

Of course the equations hold good if, instead of the two lines, we have one and the same line; the equations

$$ax^2 + by^2 + cz^2 + dw^2 = 0, \quad a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0,$$

considering therein x^2, y^2, z^2, w^2 as coordinates, may be regarded as the equations of a line; and thus the points $(x_1^2, y_1^2, z_1^2, w_1^2)$, &c., will be four points on a line. And we have thus

$$\frac{y_1^2 z_2^2 - y_2^2 z_1^2}{x_1^2 w_2^2 - x_2^2 w_1^2} = \frac{y_3^2 z_4^2 - y_4^2 z_3^2}{x_3^2 w_4^2 - x_4^2 w_3^2}, \quad \&c.,$$

equations which are, by means of the foregoing set, converted into

$$\frac{y_1 z_2 + y_2 z_1}{x_1 w_2 + x_2 w_1} = \frac{y_3 z_4 + y_4 z_3}{x_3 w_4 + x_4 w_3}, \quad \frac{z_1 x_2 + z_2 x_1}{y_1 w_2 + y_2 w_1} = \frac{z_3 x_4 + z_4 x_3}{y_3 w_4 + y_4 w_3}, \quad \frac{x_1 y_2 + x_2 y_1}{z_1 w_2 + z_2 w_1} = \frac{x_3 y_4 + x_4 y_3}{z_3 w_4 + z_4 w_3}.$$

If for x, y, z, w we write $s, c, d, 1$, then the equations are

$$\frac{c_1 d_2 - c_2 d_1}{s_1 - s_2} = \frac{c_3 d_4 - c_4 d_3}{s_3 - s_4}, \quad \frac{d_1 s_2 - d_2 s_1}{c_1 - c_2} = \frac{d_3 s_4 - d_4 s_3}{c_3 - c_4}, \quad \frac{s_1 c_2 - s_2 c_1}{d_1 - d_2} = \frac{s_3 c_4 - s_4 c_3}{d_3 - d_4},$$

$$\frac{c_1 d_2 + c_2 d_1}{s_1 + s_2} = \frac{c_3 d_4 + c_4 d_3}{s_3 + s_4}, \quad \frac{d_1 s_2 + d_2 s_1}{c_1 + c_2} = \frac{d_3 s_4 + d_4 s_3}{c_3 + c_4}, \quad \frac{s_1 c_2 + s_2 c_1}{d_1 + d_2} = \frac{s_3 c_4 + s_4 c_3}{d_3 + d_4},$$

where s_1, c_1, d_1 are the sn, cn and dn of u_1 , &c.; and where the relation between the arguments is $u_1 + u_2 = u_3 + u_4$.

In particular, if $u_4 = 0$, we have $s_4, c_4, d_4 = 0, 1, 1$; and then writing u for u_3 , and consequently s, c, d for s_3, c_3, d_3 , the relation between the arguments is $u = u_1 + u_2$; and we have

$$\frac{c_1 d_2 - c_2 d_1}{s_1 - s_2} = \frac{c - d}{s}, \quad \frac{d_1 s_2 - d_2 s_1}{c_1 - c_2} = \frac{s}{1 - c}, \quad \frac{s_1 c_2 - s_2 c_1}{d_1 - d_2} = \frac{s}{d - 1},$$

$$\frac{c_1 d_2 + c_2 d_1}{s_1 + s_2} = \frac{c + d}{s}, \quad \frac{d_1 s_2 + d_2 s_1}{c_1 + c_2} = \frac{s}{1 + c}, \quad \frac{s_1 c_2 + s_2 c_1}{d_1 + d_2} = \frac{s}{d + 1}.$$

The last two pairs give

$$\frac{1 + c}{1 - c} = \frac{(d_1 s_2 - d_2 s_1)(c_1 + c_2)}{(d_1 s_2 + d_2 s_1)(c_1 - c_2)}, \quad \frac{d + 1}{d - 1} = \frac{(s_1 c_2 - s_2 c_1)(d_1 + d_2)}{(s_1 c_2 + s_2 c_1)(d_1 - d_2)},$$

that is,

$$c = \frac{s_1 c_1 d_2 - s_2 c_2 d_1}{s_1 c_2 d_2 - s_2 c_1 d_1}, \quad d = \frac{s_1 d_1 c_2 - s_2 d_2 c_1}{s_1 c_2 d_2 - s_2 c_1 d_1}.$$

These last values give

$$c - d = \frac{(s_1 + s_2)(c_1 d_2 - c_2 d_1)}{s_1 c_2 d_2 - s_2 c_1 d_1}, \quad c + d = \frac{(s_1 - s_2)(c_1 d_2 + c_2 d_1)}{s_1 c_2 d_2 - s_2 c_1 d_1};$$

and then, from the given value of either $\frac{c-d}{s}$ or $\frac{c+d}{s}$, we obtain

$$s = \frac{s_1^2 - s_2^2}{s_1 c_2 d_2 - s_2 c_1 d_1};$$

viz. the resulting equations thus are

$$s = \frac{s_1^2 - s_2^2}{s_1 c_2 d_2 - s_2 c_1 d_1}, \quad c = \frac{s_1 c_1 d_2 - s_2 c_2 d_1}{s_1 c_2 d_2 - s_2 c_1 d_1}, \quad d = \frac{s_1 d_1 c_2 - s_2 d_2 c_1}{s_1 c_2 d_2 - s_2 c_1 d_1},$$

which are one of the four sets given (p. 63) of my *Elliptic Functions*; it may be noticed that they have the advantage of not containing k explicitly, and the disadvantage of becoming vanishing fractions for $u_1 = u_2$. To obtain the ordinary forms, we have only to multiply the numerators and denominators each by $s_1 c_2 d_2 + s_2 c_1 d_1$; the denominator thus becomes $= (s_1^2 - s_2^2)(1 - k^2 s_1^2 s_2^2)$, and each of the numerators has, or acquires, the factor $s_1^2 - s_2^2$, so that this factor divides out.