833.

ON A FORMULA IN ELLIPTIC FUNCTIONS.

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WRITING s, c, d for the sn, cn, and dn of an argument u, and so in other cases: we have s, c, d for the coordinates of a point on the quadriquadric curve $x^2+y^2=1$, $z^2+k^2x^2=1$. Applying Abel's theorem to this curve, it appears that, if $u_1+u_2+u_3+u_4=0$, the corresponding points are in a plane; that is, the elliptic functions satisfy the relation

$$\begin{vmatrix} s_1, & c_1, & d_1, & 1 \\ s_2, & c_2, & d_2, & 1 \\ s_3, & c_3, & d_3, & 1 \\ s_4, & c_4, & d_4, & 1 \end{vmatrix} = 0.$$

This may be written

$$\begin{aligned} &(s_2-s_1)(c_3d_4-c_4d_3)+(s_4-s_3)(c_1d_2-c_2d_1) \\ &+(c_2-c_1)(d_3s_4-d_4s_3)+(c_4-c_3)(d_1s_2-d_2s_1) \\ &+(d_2-d_1)(s_3c_4-s_4c_3)+(d_4-d_3)(s_1c_2-s_2c_1)=0; \end{aligned}$$

and it may be shown that each of the three lines is, in fact, separately $=0$.

This appears from the following three formulæ:

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{cn}(u_1+u_2)-\operatorname{dn}(u_1+u_2)}=\frac{s_1-s_2}{c_1d_2-c_2d_1},$$

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{cn}(u_1+u_2)+1}=\frac{c_1-c_2}{d_1s_2-d_2s_1},$$

$$\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{dn}(u_1+u_2)+1}=\frac{-\frac{1}{k^2}(d_1-d_2)}{s_1c_2-s_2c_1},$$

which are themselves at once deducible from formulæ given, p. 63, of my *Elliptic Functions*, and which may be written

$$\operatorname{sn}(u_1+u_2) = s_1^2 - s_2^2 = -(c_1^2 - c_2^2) = -\frac{1}{k^2}(d_1^2 - d_2^2), \quad \div (s_1c_2d_2 - s_2c_1d_1),$$

$$\operatorname{cn}(u_1+u_2) = s_1c_1d_2 - s_2c_2d_1, \quad \div \quad ,,$$

$$\operatorname{dn}(u_1+u_2) = s_1d_1c_2 - s_2d_2c_1, \quad \div \quad ,,$$

In fact, the numerators of $\operatorname{cn}(u_1+u_2) - \operatorname{dn}(u_1+u_2)$, $\operatorname{cn}(u_1+u_2)+1$, $\operatorname{dn}(u_1+u_2)+1$ thus become $=(s_1+s_2)(c_1d_2-c_2d_1)$, $-(c_1+c_2)(d_1s_2-d_2s_1)$, $(d_1+d_2)(s_1c_2-s_2c_1)$ respectively: so that, taking the numerator of $\operatorname{sn}(u_1+u_2)$ successively under its three forms, we have by division the formulæ in question. And then, if $u_1+u_2 = -(u_3+u_4)$, the functions on the left-hand side become, with only a change of sign, the like functions of u_3+u_4; and we thence have the required equations

$$\frac{s_1-s_2}{c_1d_2-c_2d_1} = -\frac{s_3-s_4}{c_3d_4-c_4d_3}, \text{ \&c.}$$