832.

NOTE ON AN APPARENT DIFFICULTY IN THE THEORY OF CURVES, WHEN THE COORDINATES OF A POINT ARE GIVEN AS FUNCTIONS OF A VARIABLE PARAMETER.

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SUPPOSE that the homogeneous coordinates x, y, z are given as proportional to the following functions of a parameter λ,

$$x : y : z = u + \alpha \sqrt{(\Omega)},\quad v + \beta \sqrt{(\Omega)},\quad w + \gamma \sqrt{(\Omega)},$$

where u, v, w are linear functions, Ω a cubic function, of the parameter. For the intersections of the curve with the arbitrary line $Ax + By + Cz = 0$, we have

$$Au + Bv + Cw + (A\alpha + B\beta + C\gamma) \sqrt{(\Omega)} = 0,$$

that is,

$$(Au + Bv + Cw)^2 - (A\alpha + B\beta + C\gamma)^2 \,\Omega = 0,$$

a cubic equation in λ; and the curve is thus a cubic. For the value $\lambda = \infty$ we have $x : y : z = \alpha : \beta : \gamma$, or the point (α, β, γ) is a point of the curve.

Suppose now that the line $Ax + By + Cz = 0$ is an arbitrary line through the point (α, β, γ); viz. let the coefficients A, B, C satisfy the relation $A\alpha + B\beta + C\gamma = 0$; the equation for the determination of λ becomes

$$(Au + Bv + Cw)^2 = 0,$$

which equation has two equal roots, suppose $\lambda = \lambda_0$; and the meaning of this is not at once obvious.

Observe that more properly there is a root $\lambda = \infty$ which has dropped out, and that the roots are $\lambda = \infty$, $\lambda = \lambda_0$, $\lambda = \lambda_0$. The root $\lambda = \infty$ gives the point (α, β, γ), which is of course one of the intersections of the line with the curve. The two roots λ_0 give *not the same intersection* but two different intersections of the line with the curve; the line being in fact a line through the point (α, β, γ) of the curve, and which besides meets the curve in two distinct points.

To see how this is, observe that, in the general case where $A\alpha+B\beta+C\gamma$ is not $=0$, we have λ determined by a cubic equation as above; and then taking λ equal to any root of this equation, we have further

$$Au+Bv+Cw+(A\alpha+B\beta+C\gamma)\sqrt{(\Omega)}=0,$$

viz. the value of $\sqrt{(\Omega)}$ is hereby uniquely determined; and to each of the three values of λ, $\sqrt{(\Omega)}$, there corresponds a determinate point (x, y, z).

But suppose now $A\alpha+B\beta+C\gamma=0$, and λ determined by the equation

$$(Au+Bv+Cw)^2=0,$$

giving $\lambda=\lambda_0$, as above. There is no longer an equation for the unique determination of $\sqrt{(\Omega)}$, and to the value $\lambda=\lambda_0$, there correspond the two values $\sqrt{(\Omega_0)}$, $-\sqrt{(\Omega_0)}$ of the radical: and thus to the two roots $\lambda=\lambda_0$, $\lambda=\lambda_0$ correspond the two different points

$$x : y : z=u_0+\alpha\sqrt{(\Omega_0)} : v_0+\beta\sqrt{(\Omega_0)} : w_0+\gamma\sqrt{(\Omega_0)};$$

and

$$x : y : z=u_0-\alpha\sqrt{(\Omega_0)} : v_0-\beta\sqrt{(\Omega_0)} : w_0-\gamma\sqrt{(\Omega_0)}.$$

It is to be added that the point (α, β, γ) is an inflexion on the curve. Write for a moment

$$u, v, w=a\lambda+f, b\lambda+g, c\lambda+h,$$

and let A, B, C be determined by the conditions

$$A\alpha+B\beta+C\gamma=0,$$
$$Aa+Bb+Cc=0.$$

Then the equation for the determination of λ becomes $(Af+Bg+Ch)^2=0$, viz. the left-hand is a mere constant, or there are the three equal roots $\lambda=\infty$; the intersections with the curve are thus the point (α, β, γ) three times; hence this point is an inflexion, the tangent being $Ax+By+Cz=0$. The second of the two equations may be written

$$Au_\infty+Bv_\infty+Cw_\infty=0.$$

Let λ_1 be one of the roots of the equation $\Omega=0$; u_1, v_1, w_1 the corresponding values of u, v, w, and let A, B, C, be determined by the conditions

$$A\alpha+B\beta+C\gamma=0,$$
$$Au_1+Bv_1+Cw_1=0.$$

The equation $(Au+Bv+Cw)^2=0$ for the intersections with the curve has the two equal roots $\lambda=\lambda_1$; and to each of these, since now $\sqrt{(\Omega_1)}=0$, there corresponds the same point $x : y : z=u_1 : v_1 : w_1$; hence the line $Ax+By+Cz=0$, or say

$$A_1x+B_1y+C_1z=0,$$

is a tangent from the inflexion. Similarly, if λ_2, λ_3 are the other two roots of the equation $\Omega=0$, we have $A_2x+B_2y+C_2z=0$, $A_3x+B_3y+C_3z=0$ for the other two tangents from the inflexion.

It would have been to some extent clearer to have represented the parameter λ as a quotient, say $\lambda=p/q$; the equations for x, y, z would then have been

$$x : y : z=(ap+fq)\sqrt{(q)}+\alpha\sqrt{(\Omega)} : (bp+gq)\sqrt{(q)}+\beta\sqrt{(\Omega)} : (cp+hq)\sqrt{(q)}+\gamma\sqrt{(\Omega)},$$

where Ω is now a homogeneous function $(p, q)^3$.