831.

SEMINVARIANT TABLES.

[From the *American Journal of Mathematics*, t. VII. (1885), pp. 59—73.]

THE present tables are not, I think, superseded by the tables A, pp. 149—163, contained in Capt. MacMahon's paper, "Seminvariants and Symmetric Functions," *American Journal of Mathematics*, t. VI. (1883), pp. 131—163. His order of the terms, though a very ingenious one, and giving rise to a most remarkable symmetry in the form of the tables, seems to me too artificial—and I cannot satisfy myself that it ought to be adopted in preference to the more simple one which I use: I attach also considerable importance to the employment of the simple letters b, c, d, e, &c. in place of the suffixed ones a_1, a_2, a_3, a_4, &c. There is, moreover, the question of the identification of the seminvariants with their expressions as non-unitary symmetric functions of the roots of the equation $1 + bx + \dfrac{cx^2}{1\,.\,2} + \text{\&c.} = 0$, which requires to be considered.

As to the form in which the tables present themselves, I remark that every seminvariant is a rational and integral function of the fundamental seminvariants

$$\begin{aligned} \mathrm{c} &= (1,\ b,\ c \between -b,\ 1)^2, \\ \mathrm{d} &= (1,\ b,\ c,\ d \between -b,\ 1)^3, \\ \mathrm{e} &= (1,\ b,\ c,\ d,\ e \between -b,\ 1)^4, \text{ \&c.,} \end{aligned}$$

viz. up to g, these are

c =	d =	e =	f =	g =	ce =	d² =	c³ =
c + 1	d + 1	e + 1	f + 1	g + 1			
b^2 − 1	bc − 3	bd − 4	be − 5	bf − 6			
	b^3 + 2	c^2	cd	ce	+ 1		
		b^2c + 6	b^2d + 10	d^2		+ 1	
		b^4 − 3	bc^2	b^2e + 15	− 1		
			b^3c − 10	bcd	− 4	− 6	
			b^5 + 4	c^3			+ 1
				b^3d − 20	+ 4	+ 4	
				b^2c^2	+ 6	+ 9	− 3
				b^4c + 15	− 9	− 12	+ 3
				b^6 − 5	+ 3	+ 4	− 1

and if to the value of g, which is of the weight 6, we join those of the products ce, d^2, c^3 of this same weight 6, as just written down, we have what is in effect a table of the asyzygetic seminvariants of the weight 6 and which I call the Crude Table. But we do not, in this way, obtain immediately the seminvariants of the lowest degrees: in fact, the only seminvariant containing g is given by the first column as a function $g - \ldots - 5b^6$ of the degree 6, whereas there is the seminvariant $g - 6bf + 15ce - 10d^2$ of the degree 2: to obtain this, we have to form a linear combination of the columns: the proper combination is $g + 15ce - 10d^2$, giving rise to the column g of the table (p. 278, $g = 6$). And similarly each other column of the same table is a linear combination of columns of the Crude Table: and so in every case. The process would be a very laborious one, and the tables were not, in fact, thus calculated; but we see very clearly in this manner the origin and meaning of the tables.

The mere inspection of the tables gives rise to several remarks. We see that each column begins with a non-unitary term (term without the letter b), and that it ends with a power-ending term (product wherein the last letter enters as a power)—thus, weight 8, the initial terms are

$$i,\quad cg,\quad df,\quad e^2,\quad c^2e,\quad cd^2,\quad c^4,$$

the finals are

$$e^2,\quad cd^2,\quad b^2d^2,\quad c^4,\quad b^2c^3,\quad b^4c^2,\quad b^8;$$

and it will be observed further that in this case the initial terms are all the non-unitary terms taken in order, and the corresponding final terms are all the power-ending terms taken also in order. The arrangement of the columns *inter se* is of course arbitrary, and they are, in fact, arranged so that the initial terms are the non-unitary terms taken in order—and this being so, then for each weight up to the weight 9 the final terms are the power-ending terms taken in order: but for each of the weights 10 and 11, there is a single deviation from this order; and for the weight 12, there are a great many deviations from the order.

The initial terms being in order, the broken line which bounds the tops of the columns forms a series of continually descending steps; and when the final terms are also in order, the case is the same with the broken line bounding the bottom of the columns: any deviation in the order of the final terms is shown by an ascending step or steps in the broken line bounding the bottom of the columns: thus in the table (p. 281, $k = 10$) the column cdf is longer than the next following one ce^2, and there is an ascending step accordingly.

It is to be remarked that any ascending step gives rise to a certain indeterminateness in a preceding column or columns: thus in the case just referred to, the column cdf might be replaced by any linear combination of itself with the column ce^2, it would still have the original initial and final terms cdf and b^4d^2 respectively. It would be possible to fix a standard form; we might, for instance, say that the column cdf should be that combination $cdf + 2ce^2$, which does not contain the leading term ce^2 of the ce^2-column: but I have not thought it worth while to attend to this.

It will be observed that, except in the case of an ascending step or steps, each column is completely determinate: we cannot with any column combine a preceding column, for this would give it a higher initial term: nor can we with it combine a succeeding column, for this would give it a lower final term. The numbers in the column may be taken to be without any common divisor, for any such divisor, if it existed, might be divided out: and the leading coefficient of the column may be taken to be positive.

I add certain subsidiary tables to enable the expression of any column in terms of the non-unitary symmetric functions of the roots of the equation $0 = 1 + bx + \frac{cx^2}{1.2} + \&c.$ These consist of left-hand tables and right-hand tables: the left-hand table for any weight is the original table for that weight, writing therein $b = 0$ and converting the columns into lines: thus weight $= 6$, we have

$$\begin{array}{llllll} \text{col. } g = & g + 15ce - 10d^2, & & & \\ \text{col. } ce = & & ce - & d^2 - & c^3, \\ \text{col. } d^2 = & & & d^2 + & 4c^3, \\ \text{col. } c^3 = & & & & c^3; \end{array}$$

viz. these are the values of the original columns writing therein $b = 0$.

The right-hand table is the table for the same weight taken from my paper "Tables of the Symmetric Functions of the Roots, to the Degree 10, for the form $1 + bx + \frac{cx^2}{1.2} + \ldots = (1 - \alpha x)(1 - \beta x)(1 - \gamma x)\ldots$," [829], writing therein $b = 0$, and giving only those lines of the table which relate to the non-unitary symmetric functions. Thus for weight 6, we have

$$\begin{array}{ll} 6\ (= \Sigma\alpha^6) & = \frac{1}{720}(-6g + 90ce + 60d^2 - 180c^3), \\ 42\ (= \Sigma\alpha^4\beta^2) & = \frac{1}{720}(\ \ 6g + 30ce - 60d^2 - 180c^3), \\ 3^2\ (= \Sigma\alpha^3\beta^3) & = \frac{1}{720}(\ \ 3g - 45ce + 60d^2 + \ \ 90c^3), \\ 2^3\ (= \Sigma\alpha^2\beta^2\gamma^2) & = \frac{1}{720}(-2g - 30ce + 20d^2), \end{array}$$

we thus have on the one side col. g, col. ce, col. d^2 and col. c^3, and on the other side the symmetric functions 6, 42, 3^2 and 2^3, each of them expressed as a linear function of g, ce, d^2 and c^3. It follows that each of the columns can be expressed as a linear function of the symmetric functions: and conversely each of the symmetric functions as a linear function of the columns: and this being done, each of the columns is to be regarded as having its complete value as a function of b and the other letters: for the columns *quâ* seminvariants are linear functions of the symmetric functions: and assuming them to be so, they can only be the linear functions determined by the foregoing process of writing $b = 0$.

The left-hand tables are carried up to $m = 12$; the right-hand only up to $k = 10$, the limit of the tables in the memoir last referred to.

SEMINVARIANT TABLES UP TO $(m=12)$.

$(1=0)$

1	+1

$(b=1)$

b	

$(c=2)$

	c
c	1
b^2	−1

$(d=3)$

	d
d	1
bc	−3
b^3	+2

$(e=4)$

	e	c^2
e	1	
bd	−4	
c^2	+3	1
b^2c		−2
b^4		+1

$(f=5)$

	f	cd
f	1	
be	−5	
cd	+2	+1
b^2d	+8	−1
bc^2	−6	−3
b^3c		+5
b^5		−2

$(g=6)$

	g	ce	d^2	c^3
g	+1			
bf	−6			
ce	+15	+1		
d^2	−10	−1	−1	
b^2e		−1		
bcd		+2	−6	
c^3		−1	+4	+1
b^3d			+4	
b^2c^2			−3	−3
b^4c				+3
b^6				−1

$(h=7)$	h	cf	de	c^2d
h	+1			
bg	−7			
cf	+9	+1		
de	−5	−1	+1	
b^2f	+12	−1		
bce	−30	−2	−3	
bd^2	+20	+4	−4	
c^2d		−1	+3	+1
b^3e		+3	+2	
b^2cd		−6	+12	−2
bc^3		+3	−9	−3
b^4d			−8	+1
b^3c^2			+6	+8
b^5c				−7
b^7				+2

$(i=8)$	i	cg	df	e^2	c^2e	cd^2	c^4
i	+1						
bh	−8						
cg	+28	+1					
df	−56	−3	+3				
e^2	+35	+2	−2	+1			
b^2g		−1					
bcf		+3	−9				
bde		−1	+1	−8			
c^2e		−3	+18	+6	+1		
cd^2		+2	−12		−1	+1	
b^3f			+6				
b^2ce			−15		−2		
b^2d^2			+10	+16	+1	−1	
bc^2d				−24	+2	−6	
c^4				+9	−1	+4	+1
b^4e					+1		
b^3cd					−2	+10	
b^2c^3					+1	−7	−4
b^5d						−4	
b^4c^2						+3	+6
b^6c							−4
b^8							+1

$(j=9)$	j	ch	dg	ef	c^2f	cde	d^3	c^3d
j	+ 1							
bi	− 9							
ch	+ 20	+ 2						
dg	− 28	− 7	+ 1					
ef	+ 14	+ 5	− 1	+ 1				
b^2h	+ 16	− 2						
bcg	− 56	+ 7	− 3					
bdf	+112	+22	− 2	− 4				
be^2	− 70	−25	+ 5	− 5				
c^2f		−27	+ 9	− 9	+2			
cde		+45	−17	+32	−5	+1		
d^3		−20	+ 8	−18	+3	−1	+ 1	
b^3g			+ 2					
b^2cf			− 6	+24	−4			
b^2de			+ 2	− 2	+5	−1		
bc^2e			+ 6	−51	+5	−3		
bcd^2			− 4	+34	−7	+5	− 9	
c^3d					+1	−1	+ 4	+ 1
b^4f				−12	+2			
b^3ce				+30	−5	+5		
b^3d^2				−20	−2	−2	+ 6	
b^2c^2d					+8	−6	+15	− 3
bc^4					−3	+3	−12	− 3
b^5e						−2		
b^4cd						+4	−24	+ 3
b^3c^3						−2	+17	+11
b^6d							+ 8	− 1
b^5c^2							− 6	−15
b^7c								+ 9
b^9								− 2

$(k=10)$	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
k	+ 1											
bj	− 10											
ci	+ 45	+1										
dh	−120	−4	+ 4									
eg	+210	+8	− 8	+ 16								
f^2	−126	−5	+ 5	+ 15	+ 1							
b^2i		−1										
bch		+4	−12									
bdg		−4	+ 4	− 64								
bef		+2	− 2	+ 54	−10							
c^2g		−4	+32	+ 48		+1						
cdf		+8	−64	− 60	+ 4	−3	+ 3					
ce^2		−5	+40		+16	+2	− 2	+ 1				
d^2e				+ 20	−12			− 1	+ 1			
b^2h			+ 8									
b^2cg			−28			−2						
b^2df			+56	+144	+16	+3	− 3					
b^2e^2			−35	−135	+ 9	−2	+ 2	− 1				
bc^2f				−108	−12	+3	− 9					
$bcde$				+180	−76	−1	+ 1	− 2	− 6			
bd^3				− 80	+48			+ 4	− 4			
c^3e					+48	−3	+18	+ 2	+ 4	+1		
c^2d^2					−32	+2	−12	− 3	+ 3	−1	+ 1	
b^4g						+1						
b^3cf						−3	+15					
b^3de						+1	− 1	+ 4	+ 4			
b^2c^2e						+3	−33	− 3	− 3	−3		
b^2cd^2						−2	+22	− 8	+24	+2	− 2	
bc^3d								+10	−34	+2	− 6	
c^5								− 3	+ 12	−1	+ 4	+ 1
b^5f							− 6					
b^4ce							+15			+3		
b^4d^2							−10		−16	−1	+ 1	
b^3c^2d									+24	−4	+16	
b^2c^4									− 9	+2	−11	
b^6e										−1		− 5
b^5cd										+2	−14	
b^4c^3										−1	+10	+10
b^7d											+ 4	
b^6c^2											− 3	−10
b^8c												+ 5
b^{10}												− 1

$(l=11)$	l	cj	di	eh	c^2h	fg	cdg	cef	d^2f	c^3f	de^2	c^2de	cd^3	c^4d
l	+ 1													
bk	− 11													
cj	+ 35	+ 2												
di	− 75	− 9	+ 1											
eh	+ 90	+14	− 2	+ 1										
fg	− 42	− 7	+ 1	− 1	+ 1	+ 1								
b^2j	+ 20	− 2												
bci	− 90	+ 9	− 3											
bdh	+240	+16		− 4										
beg	−420	−63	+ 9	− 2	− 5	− 5								
bf^2	+252	+42	− 6	+ 6	− 6	− 6								
c^2h		−30	+10	+ 3	− 16									
cdg		+70	−26	− 2	+ 58	+ 2	+ 1							
cef		−21	+ 7	− 6	+ 5	− 35	− 3	+ 1						
d^2f		−56	+24	+10	−100	−100	+ 6	− 3	+ 1					
de^2		+35	−15	− 5	+ 60	+ 60	− 4	+ 2	− 1	+ 1	+ 1			
b^3i			+ 2											
b^2ch			− 8		+ 32									
b^2dg			+ 8	+20	− 48	+ 8	− 1							
b^2ef			− 4	−18	+ 40		+ 3	− 1						
bc^2g			+ 8	−15	− 62	− 6	− 3							
$bcdf$			−16	−24	+232	+408	−30	+14	− 6					
bce^2			+10	+45	−205	−405	+27	−11	+ 3	− 3	− 3			
bd^2e				−10	+ 20	+ 20	+ 2	− 1	+ 3	− 8	− 8			
c^3f				+27	− 54	−270	+27	− 9	+ 4	−12				
c^2de				−45	+ 90	+450	−45	+14	− 6	+36	+ 6	+ 1		
cd^3				+20	− 40	−200	+20	− 6	+ 2	−18		− 1	+ 1	
b^4h					− 16									

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	c^2h	fg	cdg	cef	d^2f	c^3f	de^2	c^2de	cd^3	c^4d
b^3cg	+ 56		+ 5							
b^3df	− 112	− 288	+ 18	− 8	+ 4					
b^3e^2	+ 70	+ 270	− 23	+ 9	− 2	+ 2				
b^2c^2f		+ 216	− 27	+ 6	− 3	+ 36				
b^2cde		− 360	+ 51	− 16	+ 6	− 36	+ 96	− 2		
b^2d^3		+ 160	− 24	+ 8	− 8	+ 34	+ 16	+ 1	− 1	
bc^3e				+ 3	− 2	− 48	− 18	− 3		
bc^2d^2				− 2	+ 6	+ 18	− 24	+ 5	− 9	
c^4d					− 1	+ 3	+ 9	− 1	+ 4	+ 1
b^5g			− 2							
b^4cf			+ 6			− 36				
b^4de			− 2		− 4	+ 14	− 16	+ 1		
b^3c^2e			− 6		+ 3	+ 72	+ 12	+ 8		
b^3cd^2			+ 4		+ 8	− 78	− 48	− 7	+ 15	
b^2c^3d					− 10	+ 30	+ 72	− 5	+ 11	− 4
bc^5					+ 3	− 9	− 27	+ 3	− 12	− 3
b^6f						+ 12				
b^5ce						− 30		− 7		
b^5d^2						+ 20	+ 32	+ 2	− 6	
b^4c^2d							− 48	+ 10	− 39	+ 6
b^3c^4	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .	. . . . .	+ 18	− 5	+ 29	+ 14
b^7e								+ 2		
b^6cd								− 4	+ 32	− 4
b^5c^3								+ 2	− 23	− 26
b^8d									− 8	+ 1
b^7c^2									+ 6	+ 24
b^9c										− 11
b^{11}										+ 2

m	ck	dj	ei	fh	g^2	c^2i	cdh	ceg	d^2g	$\div 2$ cf^2	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2
1																			
12																			
66	+ 3																		
220	− 15	+ 15																	
495	+ 40	− 40	+ 1																
792	− 70	+ 70	− 4	+ 25															
462	+ 42	− 42	+ 3	− 24	+ 1														
	− 3																		
	+ 15	− 45																	
	− 25	+ 25	− 4																
	+ 30	− 30	+ 12	− 125															
	− 14	+ 18	− 8	+ 113	− 12														
	− 15	+ 150	+ 3			+ 1													
	+ 40	− 400	− 8	+ 50		− 4	+ 4												
	− 70	+ 700	− 22	+ 680	− 70	+ 8	− 8	+ 1											
	+ 42	− 420	+ 24	− 675	+ 100	− 5	+ 5	− 1	+ 2	+ 2									
			+ 24	− 570	+ 80			− 1	+ 5										
			− 36	+ 925	− 200			+ 2	− 19	− 4	+ 18								
			+ 15	− 400	+ 100			− 1	+ 12	+ 2	− 17	+ 1							
. . .		+ 30																	
		− 135				− 2													
		+ 360		+ 200		+ 4	− 4												
		− 630		− 525	+ 100	− 8	+ 8	− 1											
		+ 378		+ 336	− 64	+ 5	− 5	+ 1	− 2	− 2									
				− 150		+ 4	− 12												
				+ 350	− 200	− 4	+ 4	+ 2	− 30										
				− 105	+ 20	+ 2	− 2	− 2	+ 37	− 8	− 54								
				− 280	+ 320			− 2	+ 46	+ 16	− 72								
				+ 175	− 200			+ 2	− 49	− 4	+ 114	− 12							
					+ 100	− 4	+ 32	− 1	+ 20				+ 1						
					− 200	+ 8	− 64	+ 2	− 49	− 4	+ 54		− 3	+ 3					
					+ 125	− 5	+ 40	+ 1	− 32	+ 28	+ 162	− 18			+ 1				
								− 3	+ 91	− 44	− 342	+ 54	+ 4		− 2	+ 1			
								+ 1	− 32	+ 18	+ 135	− 27	− 2	− 2	+ 1	− 1	+ 1		
. . . .						+ 1													
						− 4	+ 20												
						+ 4	− 4		+ 20										
						− 2	+ 2		− 18	+ 12	+ 36								
						+ 4	− 60		− 15				− 3						

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c^2i	cdh	ceg	d^2g	$\div 2$ cf^2	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2	
−8	+120		−24	−24	+216		+6							
+5	−75		+45	−30	−360	+54	+2		−2					
			−10	+20	+66	−6	−6		+2	−1				
			+27	+12	−162		+3	−9						
			−45	+60	−486	−180	−8	−15	+4	−6				
			+20	−40	−252	+108	+7	+24	−4	+8	−12			
				−30	−81	+81	+1	+30	−2	+4		+1		
				+20	+54	−54	−2	−28	+2	−5	+8	−1	+1	
	−8													
	+28						+3							
	−56				−144		−3	+3						
	+35				+135	−27			+1					
					+108		−6	+24						
					−180	+108	+9	+30	−4	+10				
					+80	−64	+1	−16		−4	+8			
						−54	+2	−75	+2	−7		−4		
						+36	−12	+6	+4	−9	+30	+3	−3	
							+8	+48	−4	+14	−48	+2	−6	
							−2	+2	+1		+16	−1	+4	+
							−1	−16		−4				
							+3	−21						
							−1	−15		−4				
							−3	+60		+3		+6		
							+2			+8	−48	−3	+3	
								−40		−10	+68	−6	+22	
								+12		+3	−24	+3	−15	−
								+6						
								−15				−4		
								+10			+16	−23	−1	
											−24	+30	−30	
											+9	−3	+21	+
												+1		
												+2	+14	−
												−3	−9	
													−4	
													+3	+
														−
														+

Subsidiary Tables: $b=0$.

Left-hand: up to $(m=12)$.

Col.	c
c	+1

Col.	d
d	+1

Col.	e	c^2
e	+1	+3
c^2		+1

Col.	f	cd
f	+1	+2
cd		+1

Col.	g	ce	d^2	c^3
g	+1	+15	−10	
ce		+1	−1	−1
d^2			+1	+4
c^3				+1

Col.	h	cf	de	c^2d
h	+1	+9	−5	
cf		+1	−1	−1
de			+1	+3
c^2d				+1

Col.	l	cg	df	e^2	c^2e	cd^2	c^4
l	+1	+28	−56	+35			
cg		+1	−3	+2	−3	+2	
df			+3	−2	+18	−12	
e^2				+1	+6	.	+9
c^2e					+1	−1	−1
cd^2						+1	+4
c^4							+1

Col.	j	ch	dg	ef	c^2f	cde	d^3	c^3d
j	+1	+20	−28	+14				
ch		+2	−7	+5	−27	+45	−20	
dg			+1	−1	+9	−17	+8	
ef				+1	−9	+32	−18	
c^2f					+2	−5	+3	+1
cde						+1	−1	−1
d^3							+1	+4
c^3d								+1

Right-hand: up to $(k=10)$.

$\div 2$

	c
2	−2

$\div 6$

	d
3	−3

$\div 24$

	e	c^2
4	−4	+12
2^2	+2	+6

$\div 120$

	f	cd
5	−5	+50
32	+5	+10

$\div 720$

	g	ce	d^2	c^3
6	−6	+90	+60	−180
42	+6	+30	−60	−180
3^2	+3	−45	+60	+90
2^3	−2	−30	+20	.

$\div 5040$

	h	cf	de	c^2d
7	−7	+147	+245	−1470
52	+7	+63	−245	−630
43	+7	−147	+175	+210
32^2	−7	−63	+35	.

$\div 40320$

	i	cg	df	e^2	c^2e	cd^2	c^4
8	−8	+224	+448	+280	−3360	−4480	+5040
62	+8	+112	−448	−280	−1680	+1120	+5040
53	+8	−224	+392	−280	+3360	−3920	−5040
4^2	+4	−112	−224	+420	−1680	+2240	+2520
42^2	−8	−112	+448	−280	+1680	−1120	.
3^22	−8	+56	−392	+280	−840	+560	
2^4	+2	+56	−112	+70			

$\div 362880$

	j	ch	dg	ef	c^2f	cde	d^3	c^3d
9	−9	+324	+756	+1134	−6804	−22680	−5040	+68040
72	+9	+180	−756	−1134	−3780	+5040	+5040	+37800
63	+9	−324	+756	−1134	+6804	.	−10080	−22680
54	+9	−324	−756	+1386	−756	−2520	+5040	+7560
52^2	−9	−180	+756	−126	.	+7560	−5040	.
432	−18	+144	.	−252	+4536	−10080	+5040	.
3^3	−3	+108	−504	+378	−2268	+3780	−1680	.
32^3	+9	+180	−252	+126	.	.	.	.

Left-hand.

Col.	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
k	+1	+45	−120	+210	−126							
ci		+1	−4	+8	−5	−4	+8	−5				
dh			+4	−8	+5	+32	−64	+40				
eg				+16	−15	+48	−60	·	+20			
f^2					+1	·	+4	+16	−12	+48	−32	
c^2g						+1	−3	+2	·	−3	+2	
cdf							+3	−2	·	+18	−12	
ce^2								+1	−1	+2	−3	−3
d^2e									+1	+4	+3	+12
c^3e										+1	−1	−1
c^2d^2											+1	+4
c^5												+1

$\div 3628800$

Right-hand.

	k	ci	dh	eg	f^2	c^2g	cdf	ce^2	d^2e	c^3e	c^2d^2	c^5
10	−10	+450	+1320	+2100	+1260	−12600	−50400	−31500	−42000	+189000	+378000	−226800
82	+10	+270	−1320	−2100	−1260	−7560	+10080	+6300	+42000	+113400	+25200	−226800
73	+10	−450	+1320	−2100	−1260	+12600	−2520	+31500	−46200	−189000	+151200	+226800
64	+10	−450	−1200	+2940	−1260	−2520	+50400	−44100	−8400	+189000	−226800	−226800
5^2	+5	−225	−600	−1050	+2520	+6300	−37800	+15700	+21000	−94500	+126000	+113400
62^2	−10	−270	+1200	−420	+1260	·	−10080	+31500	−16800	−75600	−50400	
532	−20	+180	−120	+4200	−3780	−5040	+42840	−37800	+4200	+75600	−50400	
4^22	−10	+90	+1200	−2940	+1260	+12600	−30240	+6300	+8400	−37800	+25200	
43^2	−10	+450	−1320	−420	+1260	−5040	+2520	+6300	−4200	·		
42^3	+10	+270	−1200	+2100	−1260	−5040	+10080	−6300				
3^22^2	+15	+45	+720	−1890	+1260	+2520	−5040	+3150				
2^5	−2	−90	+240	−420	+252							

Left-hand, no right-hand.

Col.	l	cj	di	eh	fg	c^2h	cdg	cef	d^2f	de^2	c^3f	c^2de	cd^3	c^4d
l	+1	+35	−75	+90	−42									
cj		+2	−9	+14	−7	−30	+70	−21	−56	+35				
di			+1	−2	+1	+10	−26	+7	+24	−15				
eh				+1	−1	+3	−2	−6	+10	−5	+27	−45	+20	
fg					+1	−16	+58	+5	−100	+60	−54	+90	−40	
c^2h						+2	−7	+5	·	·	−27	+45	−20	
cdg							+1	−3	+6	−4	+27	−45	+20	
cef								+1	−3	+2	−9	+14	−6	
d^2f									+1	−1	+4	−6	+2	−1
de^2										+1	−12	+36	−18	+3
c^3f											+2	−5	+3	+1
c^2de												+1	−1	−1
cd^3													+1	+4
c^4d														+1

Col.	m	ck	dj	ei	fh	g^2	c^2i	cdh	ceg	cf^2	d^2g	def	e^3	c^3g	c^2df	c^2e^2	cd^2e	d^4	c^4e	c^3d^2	c^6
m	+1	+66	−220	+495	−792	+462															
ck		+3	−15	+40	−70	+42	−15	+40	−70	+42											
dj			+15	−40	+70	−42	+150	−400	+700	−420											
ei				+1	−4	+3	+3	−8	−22	+24	+24	−36	+15								
fh					+25	−24		+50	+680	−675	−570	+925	−400								
g^2						+1			−70	+100	+80	−200	+100	−4	+8	−5					
c^2i							+1	−4	+8	−5				−4	+8	−5					
cdh								+4	−8	+5				+32	−64	+40					
ceg									+1	−1	−1	+2	−1	−1	+2	+1	−3	+1			
cf^2										+2	+5	−19	+12	+20	−49	−32	+91	−32			
d^2g											+1	−3	+2	+4	−9	−12	+27	−10	+6	−4	
def												+18	−17		+54	+162	−342	+135	−81	+54	
e^3													+1			−18	+54	−27	+81	−54	
c^3g														+1	−3				+1	−2	−2
c^2df															+3			−2	+30	−28	+2
c^2e^2																+1	−2	+1	−2	+2	+1
cd^2e																	+1	−1	+4	−5	·
d^4																		+1		+8	+16
c^4e																			+1	−1	−1
c^3d^2																				+1	+4
c^6																					+1