826.

NOTE ON A PARTITION-SERIES.

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PROF. SYLVESTER, in his paper, "A Constructive theory of Partitions, &c.," *American Journal of Mathematics*, vol. V. (1883), p. 282, has given the following very beautiful formula

$$(1+ax)(1+ax^2)(1+ax^3)\ldots = 1+\frac{1}{1-x}(1+ax^2)\,xa+\frac{1}{1-x\,.\,1-x^2}(1+ax)(1+ax^4)\,x^5a^2$$
$$+\frac{1}{1-x\,.\,1-x^2\,.\,1-x^3}(1+ax)(1+ax^2)(1+ax^6)\,x^{12}a^3+\ldots,$$

or, as this may be written,

$$\Omega = 1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3)+\ldots\quad,$$

where

$$P=\frac{(1+ax^2)\,xa}{\mathbf{1}},\quad Q=\frac{(1+ax^4)\,x^5a^2}{\mathbf{1}\,.\,\mathbf{2}},\quad R=\frac{(1+ax^6)\,x^{12}a^3}{\mathbf{1}\,.\,\mathbf{2}\,.\,\mathbf{3}},\quad S=\frac{(1+ax^8)\,x^{22}a^4}{\mathbf{1}\,.\,\mathbf{2}\,.\,\mathbf{3}\,.\,\mathbf{4}},\ \&\text{c.},$$

the heavy figures **1**, **2**, **3**, **4**,... of the denominators being, for shortness, written to denote $1-x$, $1-x^2$, $1-x^3$, $1-x^4$,... respectively. The x-exponents 1, 5, 12, 22,... are the pentagonal numbers $\frac{1}{2}(3n^2-n)$.

To prove this, writing

$$P'=\frac{ax^2}{\mathbf{1}},\quad Q'=\frac{ax^3}{\mathbf{1}}+\frac{a^2x^7}{\mathbf{1}\,.\,\mathbf{2}},\quad R'=\frac{ax^4}{\mathbf{1}}+\frac{a^2x^9}{\mathbf{1}\,.\,\mathbf{2}}+\frac{a^3x^{15}}{\mathbf{1}\,.\,\mathbf{2}\,.\,\mathbf{3}},\quad S'=\frac{ax^5}{\mathbf{1}}+\frac{a^2x^{11}}{\mathbf{1}\,.\,\mathbf{2}}+\frac{a^3x^{18}}{\mathbf{1}\,.\,\mathbf{2}\,.\,\mathbf{3}}+\frac{a^4x^{26}}{\mathbf{1}\,.\,\mathbf{2}\,.\,\mathbf{3}\,.\,\mathbf{4}},\ \&\text{c.},$$

where the x-exponents are

$$2;\ 3,\ 3+4;\ 4,\ 4+5,\ 4+5+6;\ 5,\ 5+6,\ 5+6+7,\ 5+6+7+8;\ \&\text{c.},$$

we find without difficulty (see infrà) that

$$1+P=(1+ax)(1+P'),$$
$$1+P'+Q=(1+ax^2)(1+Q'),$$
$$1+Q'+R=(1+ax^3)(1+R'),$$
$$1+R'+S=(1+ax^4)(1+S'),\ \&c.;$$

and hence, using Ω to denote the sum

$$\Omega=1+P+Q(1+ax)+R(1+ax)(1+ax^2)+S(1+ax)(1+ax^2)(1+ax^3)+\ldots,$$

we obtain successively

$$\Omega\div(1+ax)=1+P'+Q+R(1+ax^2)+S(1+ax^2)(1+ax^3)+\ldots,$$
$$\Omega\div(1+ax)(1+ax^2)=1+Q'+R+S(1+ax^3)+T(1+ax^3)(1+ax^4)+\ldots,$$
$$\Omega\div(1+ax)(1+ax^2)(1+ax^3)=1+R'+S+T(1+ax^4)+\ldots,$$

and so on. In these equations, on the right-hand sides, the lowest exponent of x is 2, 3, 4, &c., respectively, so that in the limit the right-hand side becomes $=1$, or the final equation is $\Omega=(1+ax)(1+ax^2)(1+ax^3)\ldots$; viz. we have the series represented by Ω equal to this infinite product, which is the theorem in question.

One of the foregoing identities is

$$1+R'+S=(1+ax^4)(1+S'),$$

viz. substituting for R', S, S' their values, this is

$$1+\frac{ax^4}{1}+\frac{a^2x^9}{1.2}+\frac{a^3x^{15}}{1.2.3}+\frac{(1+ax^8)a^4x^{22}}{1.2.3.4}=(1+ax^4)\left\{1+\frac{ax^5}{1}+\frac{a^2x^{11}}{1.2}+\frac{a^3x^{18}}{1.2.3}+\frac{a^4x^{26}}{1.2.3.4}\right\},$$

viz. this equation is

$$-ax^4+\frac{ax^4-ax^5(1+ax^4)}{1}+\frac{a^2x^9-a^2x^{11}(1+ax^4)}{1.2}$$
$$+\frac{a^3x^{15}-a^3x^{18}(1+ax^4)}{1.2.3}+\frac{(1+ax^8)a^4x^{22}-a^4x^{26}(1+ax^4)}{1.2.3.4},$$

that is,

$$0=-ax^4+ax^4-\frac{a^2x^9}{1}+\frac{a^2x^9}{1}-\frac{a^3x^{15}}{1.2}+\frac{a^3x^{15}}{1.2}-\frac{a^4x^{22}}{1.2.3}+\frac{a^4x^{22}}{1.2.3}.$$

In the same way each of the other identities is proved.

Writing $a=-1$, we have Ω, $=\mathbf{1.2.3.4}\ldots$,

$$=1+P+Q.\mathbf{1}+R.\mathbf{1.2}+S.\mathbf{1.2.3}+\ldots,$$

where

$$P=-(1+x)x,\quad Q=\frac{(1+x^2)x^5}{1},\quad R=-\frac{(1+x^3)x^{12}}{1.2},\ldots$$

and therefore

$$\mathbf{1.2.3.4}\ldots = 1-(1+x)x+(1+x^2)x^5-(1+x^3)x^{12}+\ldots,$$

which is Euler's theorem.

It might appear that the identities used in the proof would also, for this particular value $a=-1$, lead to interesting theorems; but this is found *not* to be the case: we have

$$P'=\frac{-x^2}{\mathbf{1}},\quad Q'=\frac{-x^3}{\mathbf{1}}+\frac{x^7}{\mathbf{1.2}},\quad R'=\frac{-x^4}{\mathbf{1}}+\frac{x^9}{\mathbf{1.2}}-\frac{x^5}{\mathbf{1.2.3}},\ \text{\&c.},$$

but the expressions in terms of these quantities for the products $\mathbf{2.3.4}\ldots$, $\mathbf{3.4}\ldots$, &c., contain denominator factors, and are thus altogether without interest; we have, for example,

$$\mathbf{2.3.4}\ldots = 1+\frac{-x^2+x^5+x^7}{\mathbf{1}}-\frac{(1+x^3)x^{12}}{\mathbf{1}}+\text{\&c.},$$

which is, with scarcely a change of form, the expression obtained from that of the original product $\mathbf{1.2.3.4}\ldots$, by division by $\mathbf{1}$, $=1-x$. And similarly as regards the products $\mathbf{3.4}\ldots$, &c.

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