822.

ON ASSOCIATIVE IMAGINARIES.

[From the *Johns Hopkins University Circulars*, No. 15 (1882), pp. 211, 212.]

THE imaginaries (x, y) defined by the equations

$$x^2 = ax + by,$$
$$xy = cx + dy,$$
$$yx = ex + fy,$$
$$y^2 = gx + hy,$$

will not be in general associative: to make them so, we must have 8 double relations corresponding to the combinations x^3, x^2y, xyx, xy^2, yx^2, yxy, y^2x, y^3 respectively, viz. the first of these gives $x.x^2 - x^2.x = 0$, that is, $0 = x(ax+by) - (ax+by)x$, $= b(xy - yx)$, $= b[(c-e)x + (d-f)y]$; that is, $0 = b(c-e)$ and $0 = b(d-f)$: and similarly for each of the other terms. We thus obtain apparently 16, but really only 12, relations between the 8 coefficients a, b, c, d, e, f, g, h, viz. the relations so obtained are

$$b(e-c) = 0 \text{ (twice)},\quad b(d-f) = 0,\quad g(c-e) = 0,\quad g(f-d) = 0 \text{ (twice)},$$
$$bg - ed = 0 \text{ (twice)},\quad bg - ef = 0 \text{ (twice)},$$
$$c^2 + dg - ag - ch = 0,\quad d^2 + bc - ad - bh = 0,\quad e^2 + fg - ag - eh = 0,\quad f^2 + be - af - bh = 0,$$
$$a(c-e) - cf + de = 0,\quad h(f-d) - cf + de = 0.$$

From the first four equations it appears that either $b = 0$ and $g = 0$, or else $c = e$ and $d = f$: for brevity, I attend only to the latter case, giving the commutative system

$$x^2 = ax + by,$$
$$xy = yx = cx + dy,$$
$$y^2 = gx + hy.$$

In order that this may be associative, we must still have the relations

$$bg = cd,$$
$$c^2 + dg - ag - ch = 0,$$
$$d^2 + bc - ad - bh = 0,$$

which are all three of them satisfied by $g = \frac{cd}{b}$, $h = \frac{d^2 + bc - ad}{b}$, viz. we thus have the associative and commutative system

$$x^2 = ax + by,$$
$$xy = yx = cx + dy,$$
$$y^2 = \frac{cd}{b}x + \frac{d^2 + bc - ad}{b}y.$$

I did not perceive how to identify this system with any of the double algebras of B. Peirce's Linear Associative Algebra, see pp. 120—122 of the Reprint, *American Journal of Mathematics*, t. IV. (1881); but it has been pointed out to me by Mr C. S. Peirce that my system, in the general case $ad - bc$ not $= 0$, is expressible as a mixture of two algebras of the form (a_1), see p. 120 (*l.c.*); whereas if $ad - bc = 0$, it is reducible to the form (c_2), see p. 122 (*l.c.*). The object of the present Note is to exhibit in the simple case of two imaginaries the whole system of relations which must subsist between the coefficients in order that the imaginaries may be associative; that is, the system of equations which are solved implicitly by the establishment of the several multiplication tables of the memoir just referred to.