819.

ON TWO CASES OF THE QUADRIC TRANSFORMATION BETWEEN TWO PLANES.

[From the *Johns Hopkins University Circulars*, No. 13 (1882), pp. 178, 179.]

SEEKING for the coordinates x_3, y_3, z_3 of the third point of intersection of the cubic curve $x^3+y^3+z^3+6lxyz=0$ by the line through any two points (x_1, y_1, z_1), (x_2, y_2, z_2) on the curve, the expressions present themselves in the form

$$x_3 : y_3 : z_3 = P+2lA : Q+2lB : R+2lC,$$

where

$$P = x_1y_1y_2^2 + z_1x_1z_2^2 - y_1^2x_2y_2 - z_1^2z_2x_2, \quad A = x_1^2y_2z_2 - y_1z_1x_2^2,$$
$$Q = y_1z_1z_2^2 + x_1y_1x_2^2 - z_1^2y_2z_2 - x_1^2x_2y_2, \quad B = y_1^2z_2x_2 - z_1x_1y_2^2,$$
$$R = z_1x_1x_2^2 + y_1z_1y_2^2 - x_1^2z_2x_2 - y_1^2y_2z_2, \quad C = z_1^2x_2y_2 - x_1y_1z_2^2;$$

but it is known that, in virtue of

$$U_1 = x_1^3+y_1^3+z_1^3+6lx_1y_1z_1 = 0, \quad U_2 = x_2^3+y_2^3+z_2^3+6lx_2y_2z_2 = 0,$$

which connect the coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2), we have $P:Q:R = A:B:C$*, so that the coordinates (x_3, y_3, z_3) of the third point of intersection may be expressed indifferently in the two forms

$$x_3 : y_3 : z_3 = P : Q : R, \quad \text{and} \quad x_3 : y_3 : z_3 = A : B : C.$$

But these considered irrespectively of the equations $U_1=0$, $U_2=0$, are distinct formulæ, each of them separately establishing a correspondence between the three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), or if we regard one of these points as a fixed point, then a correspondence between the remaining two points, or if we consider these as belonging each to its own plane, then a correspondence between two planes.

* See Sylvester on Rational Derivation of Points on Cubic Curves, *Amer. Jour. of Math.* vol. III. p. 62.

Writing for convenience (a, b, c) for the coordinates of the fixed point, and (x_1, y_1, z_1), (x_2, y_2, z_2) for those of the other two points, the formulæ with A, B, C give thus the correspondence

$$x_2 : y_2 : z_2 = bcx_1^2 - a^2y_1z_1 : cay_1^2 - b^2z_1x_1 : abz_1^2 - c^2x_1y_1,$$

which is the first of the two cases in question. These equations give reciprocally

$$x_1 : y_1 : z_1 = bcx_2^2 - a^2y_2z_2 : cay_2^2 - b^2z_2x_2 : abz_2^2 - c^2x_2y_2,$$

or the correspondence is a (1, 1) quadric correspondence.

The formulæ with P, Q, R give in like manner

$$x_2 : y_2 : z_2 = a(ax_1^2 + by_1^2 + cz_1^2) - x_1(a^2x_1 + b^2y_1 + c^2z_1), \text{ \&c.},$$

or if for shortness

$$\Omega_1 = ax_1^2 + by_1^2 + cz_1^2, \quad \Theta_1 = a^2x_1 + b^2y_1 + c^2z_1,$$

then

$$x_2 : y_2 : z_2 = a\Omega_1 - x_1\Theta_1 : b\Omega_1 - y_1\Theta_1 : c\Omega_1 - z_1\Theta_1,$$

which is the second of the two cases. We have reciprocally

$$x_1 : y_1 : z_1 = a\Omega_2 - x_2\Theta_2 : b\Omega_2 - y_2\Theta_2 : c\Omega_2 - z_2\Theta_2,$$

where

$$\Omega_2 = ax_2^2 + by_2^2 + cz_2^2, \quad \Theta_2 = a^2x_2 + b^2y_2 + c^2z_2,$$

and the correspondence is thus in this case also a (1, 1) quadric correspondence.