815.

THE BINOMIAL EQUATION $x^p - 1 = 0$; QUINQUISECTION. SECOND PART.

In the paper, "The binomial equation $x^p - 1 = 0$; quinquisection," *Proc. Lond. Math. Soc.*, t. XII. (1881), pp. 15, 16, [764], I considered for an exponent $p = 5n + 1$, the five periods X, Y, Z, W, T connected by the equations

	X,	Y,	Z,	W,	T
$X^2 =$	a,	b,	c,	d,	e
$XY =$	f,	g,	h,	i,	j
$XZ =$	k,	l,	m,	n,	o,

and the equations deduced from these by cyclical permutations of the periods and of the coefficients of each set; but I did not obtain completely even the linear relations connecting the coefficients. I since found, by induction from the examples given in the Table 1, that the coefficients could be expressed linearly in terms of the linearly independent integer numbers α, β, f, k as follows: viz. introducing for convenience the new number θ, such that

$$\alpha + \beta + \theta = \tfrac{1}{5}(p - 1),$$

then the expressions in question are

$$a, b, c, d, e = -1 - 2\theta + \alpha + \beta, \quad -\theta - \alpha - \beta + f, \quad -\theta - \alpha - \beta + k, \quad -\alpha - 2\beta - k, \quad -2\alpha - \beta - f,$$
$$f, g, h, i, j = \qquad f, \qquad\qquad \theta - \alpha - f, \qquad\qquad \alpha, \qquad\qquad \beta, \qquad\qquad \alpha,$$
$$k, l, m, n, o = \qquad k, \qquad\qquad \alpha, \qquad\qquad \theta - \beta - k, \qquad\qquad \beta, \qquad\qquad \beta,$$

and I found further that, substituting these values of the coefficients in the 20 quadric relations referred to in the former paper, the 20 relations reduced themselves to two equations only, viz. these were

$$\theta(-2\alpha + \beta + k) + 3\alpha^2 - \beta^2 \quad + \quad \alpha(f - k - 1) - \beta f + f^2 - 2fk = 0,$$
$$-\theta^2 + \theta(3\beta + 2k + f) \quad + \quad \alpha^2 - \alpha\beta - 3\beta^2 - \alpha k + \beta(1 - f - k) - k^2 - 2fk = 0.$$

The final result thus is that the coefficients are expressed as functions of the five numbers α, β, f, k, θ, connected by the linear equation $\alpha + \beta + \theta = \frac{1}{5}(p-1)$, and the two quadric equations. I remark that formulæ equivalent to these were obtained and proved by Mr F. S. Carey in his Trinity Fellowship Dissertation, 1884; viz. writing $n = \frac{1}{5}(p-1)$, his formulæ were

$$a,\ b,\ c\ ,\ d,\ e = \alpha - n,\ \ \beta - n,\ \ \gamma - n,\ \ \delta - n,\ \ \epsilon - n,$$
$$f,\ g,\ h\ ,\ i\ ,\ j = \beta,\ \ \ \ \ \ \epsilon,\ \ \ \ \ \ \rho,\ \ \ \ \ \ \sigma,\ \ \ \ \ \ \rho,$$
$$k,\ l,\ m,\ n,\ o = \gamma,\ \ \ \ \ \ \rho,\ \ \ \ \ \ \delta,\ \ \ \ \ \ \sigma,\ \ \ \ \ \ \sigma,$$

with the three linear relations

$$\alpha + \beta + \gamma + \delta + \epsilon = n - 1,$$
$$\beta + \epsilon + 2\rho + \sigma = n,$$
$$\gamma + \delta + \rho + 2\sigma = n,$$

and the two quadric relations

$$\delta^2 + \gamma^2 + 2\sigma\alpha + (\rho - \sigma)(\delta + \gamma) - 2\rho(\rho + \sigma) = (\delta - \gamma)(\beta - \epsilon),$$
$$\beta^2 + \epsilon^2 + 2\rho\alpha + (\sigma - \rho)(\beta + \epsilon) - 2\sigma(\rho + \sigma) = (\gamma - \delta)(\beta - \epsilon),$$

the coefficients being thus expressed in terms of the seven numbers α, β, γ, δ, ϵ, ρ, σ connected by five equations. The equivalence of the two sets of formulæ may be shown without difficulty.

To the Table 2 of the Quintic Equations, given in the paper, may be added the following result from Legendre's *Théorie des Nombres*, Ed. 3, t. II., p. 213,

p	η^5	η^4	η^3	η^2	η	1	
641	1	$+1$	-256	-564	$+5238$	-5120	$= 0,$

calculated by him for the isolated case $p = 641$.