

806.

DETERMINATION OF THE ORDER OF A SURFACE.

[From the *Messenger of Mathematics*, vol. XII. (1883), pp. 29—32.]

[ON p. lxx of the Prolegomena to C. Taylor's *Introduction to the Geometry of Conics* (1881) occurs the following passage:

“*Proof and extension of Newton's Descriptio Organica.*

“Let two angles AOB and $A\omega B$ of given magnitudes turn about O and ω respectively, and let the intersection A trace a curve of the n th order. For a given position of the arm OB there are n positions of A and therefore n of B . When OB is in the position $O\omega$ the n B 's coincide with ω , which is therefore an n -fold point on the locus of B , as is also the point O ; and since any line through O (or ω) meets the locus of B in n other points, the locus is of the order $2n$. Its order is the same when $A\omega B$ is a zero-angle or straight line.

“Let a given trihedral angle $O(ABC)$ —or a plane OBC and a line OA rigidly attached to it—turn about O , and let a variable plane through a fixed point ω meet OA in A and the plane OBC in BC ; then if the line BC describes a ruled surface of the order n the point A describes a surface of the order $4n$.”

And in a foot-note it is stated that the author is indebted to Professor Cayley for the determination of the order of this surface. The following paper contains Professor Cayley's determination, which was communicated by him to Mr Taylor.]*

LEMMA. Take (a, b, c, f, g, h) the six coordinates of a line; these are connected by the equation

$$af + bg + ch = 0.$$

[* *l.c.*, p. 29.]

In order that the line may belong to a ruled surface, we must have between the coordinates three more equations, say these are

$$F(a, b, c, f, g, h) = 0,$$

$$G(a, \quad, \quad) = 0,$$

$$H(a, \quad, \quad) = 0,$$

of the orders p, q, r respectively; then the scroll is of the order $n = 2pqr$.

For expressing that the line meets an arbitrary line (a', b', c', f', g', h') , we have the linear relation

$$f'a + g'b + h'c + a'f + b'g + c'h = 0;$$

the five relations determine the ratios $a : b : c : f : g : h$, and the number of systems of values is = product of orders $= 2 \cdot p \cdot q \cdot r \cdot 1 = 2pqr$, viz. this is the number of lines meeting the arbitrary line; or, what is the same thing, it is = order of ruled surface.

Consider now a trihedral angle $OABC$ rotating about a fixed point O which may be taken for the origin, and consider a fixed point ω . Let the lines OB, OC each meet a line L , and the plane ωL intersect OA in a point P ; then it is to be shown that, if L is a line of a ruled surface of the order n , the locus of P is a surface of the order $4n$.

Observe that, for a given position of the line L , the position of the lines OB, OC is not determinate, but that the angle BOC has any position at pleasure in the plane OL , or say it rotates round the line ON which is the normal at O to the plane OL ; the line OA therefore also rotates about ON , being always inclined to it at a determinate angle θ , or the locus of OA is a right cone axis ON and semi-aperture $= \theta$. The points P are the intersections of the several lines of the cone with the fixed plane ωL , or say that (for the given position of L) the locus of P is the conic C which is the intersection of the cone in question by the plane ωL . And then varying the position of L , the required surface is the locus of the corresponding conics C .

Take now

$$Ax + By + Cz + D = 0, \quad A'x + B'y + C'z + D' = 0,$$

for the equations of a particular line L ; then writing

$$AD - A'D, \quad BD - B'D, \quad CD - C'D, \quad BC' - B'C, \quad CA' - C'A, \quad AB' - A'B = (a, b, c, f, g, h)$$

respectively, these are the six coordinates of the line L .

The equation of the plane OL is

$$(AD - A'D)x + (BD - B'D)y + (CD - C'D)z = 0,$$

that is,

$$ax + by + cz = 0;$$

and the equations of the normal ON are therefore

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c};$$

we have therefore for the cone

$$\frac{ax + by + cz}{\sqrt{(x^2 + y^2 + z^2)} \sqrt{(a^2 + b^2 + c^2)}} = \cos \theta;$$

or if, for convenience, $\cos^2 \theta = k$, then the equation of the cone is

$$(ax + by + cz)^2 - k(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = 0;$$

say this is

$$\{a^2 - k(a^2 + b^2 + c^2), b^2 - k(a^2 + b^2 + c^2), c^2 - k(a^2 + b^2 + c^2), bc, ca, ab\} (x, y, z)^2 = 0 \dots (1).$$

Taking then (x_0, y_0, z_0) for the coordinates of the point ω , the equation of the plane ωL is

$$\frac{Ax + By + Cz + D}{Ax_0 + By_0 + Cz_0 + D} - \frac{A'x + B'y + C'z + D'}{A'x_0 + B'y_0 + C'z_0 + D'} = 0,$$

viz. it is

$$(BC' - B'C)(yz_0 - y_0z) + \dots + (AD' - A'D)(x - x_0) + \dots = 0,$$

that is,

$$f(yz_0 - y_0z) + g(zx_0 - z_0x) + h(xy_0 - x_0y) + a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

or say

$$x(hy_0 - gz_0 + a) + y(-hx_0 + fz_0 + b) + z(gx_0 - fy_0 + c) + (-ax_0 - by_0 - cz_0) = 0 \dots (2),$$

viz. (1) and (2) are the equations of the conic C , and the coordinates (a, b, c, f, g, h) satisfy of course the equation

$$af + bg + ch = 0 \dots \dots \dots (3).$$

Considering now the line L as belonging to a ruled surface, the coordinates (a, \dots) satisfy as before three equations

$$F(a, b, c, f, g, h) = 0 \dots \dots \dots (4),$$

$$G(\quad, \quad, \quad) = 0 \dots \dots \dots (5),$$

$$H(\quad, \quad, \quad) = 0 \dots \dots \dots (6),$$

of the orders p, q, r respectively, and we can from the six equations eliminate a, b, c, f, g, h . The resulting equation $\nabla = 0$ contains the coefficients of (1) in the order $1.2.p.q.r = 2pqr$ (which is the product of the orders of the other 5 equations), and the coefficients of (2) in the order $2.2.p.q.r = 4pqr$, (which is the product of the orders of the other 5 equations). But the coefficients of (1) being quadric functions of (x, y, z) , and those of (2) being linear functions of (x, y, z) , the aggregate order in (x, y, z) is

$$2.2pqr + 1.4pqr, = 8pqr;$$

or, since the order of the ruled surface is $n, = 2pqr$, the order of the locus is $= 4n$; which is the above-stated theorem.