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NOTE ON CAPTAIN MACMAHON'S PAPER, "ON THE DIFFERENTIAL EQUATION $X^{-\frac{2}{3}}dx + Y^{-\frac{2}{3}}dy + Z^{-\frac{2}{3}}dz = 0$."

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 182—184.]

IN general, if $f, = (x, y, 1)^3, = 0$ be the equation of a cubic curve, and if

$$d\omega = \frac{dx}{\frac{df}{dy}}, \quad = \frac{-dy}{\frac{df}{dx}},$$

then if 1, 2, 3 are the intersections of the curve by an arbitrary right line, the coordinates of these points being (x_1, y_1), (x_2, y_2), (x_3, y_3) respectively, we have, by Abel's theorem,

$$d\omega_1 + d\omega_2 + d\omega_3 = 0,$$

viz. this is the differential relation corresponding to the integral relation which expresses that the three points are the intersections of the cubic curve by a right line, or say to the integral equation

$$\nabla, = \begin{vmatrix} y_1, & y_2, & y_3 \\ x_1, & x_2, & x_3 \\ 1, & 1, & 1 \end{vmatrix}, \quad = 0,$$

in which equation y_1, y_2, y_3 are regarded as functions of x_1, x_2, x_3 respectively, given by means of the equations $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ which express that the points are on the cubic curve. See my "Memoir on the Abelian and Theta Functions," [819].

In particular, if the equation of the curve is

$$f, \quad = \tfrac{1}{3}\{y^3 - (A + 3Bx + 3Cx^2 + Dx^3)\}, \quad = \tfrac{1}{3}(y^3 - X), \quad = 0,$$

then

$$d\omega = \frac{dx}{y^2}, \quad = \frac{dx}{X^{\frac{2}{3}}}:$$

and corresponding to the differential relation

$$X_1^{-\frac{2}{3}}\,dx_1 + X_2^{-\frac{2}{3}}\,dx_2 + X_3^{-\frac{2}{3}}\,dx_3 = 0,$$

we have the integral relation

$$\nabla, \ = \begin{vmatrix} X_1^{\frac{1}{3}}, & X_2^{\frac{1}{3}}, & X_3^{\frac{1}{3}} \\ x_1\ , & x_2\ , & x_3 \\ 1\ , & 1\ , & 1 \end{vmatrix} = 0,$$

viz. this last equation, as containing no arbitrary constant, is a particular integral of the differential equation.

If instead of x_1, x_2, x_3 we write x, y, z, then we have

$$\nabla = \begin{vmatrix} X^{\frac{1}{3}}, & Y^{\frac{1}{3}}, & Z^{\frac{1}{3}} \\ x\ , & y\ , & z \\ 1\ , & 1\ , & 1 \end{vmatrix}, \ = 0,$$

as a particular integral of the differential equation

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0.$$

To rationalize the integral equation, write

$$\alpha,\ \beta,\ \gamma = y - z,\ z - x,\ x - y \ \text{(so that } \alpha + \beta + \gamma = 0\text{)},$$

the equation is

$$\alpha X^{\frac{1}{3}} + \beta Y^{\frac{1}{3}} + \gamma Z^{\frac{1}{3}} = 0\,;$$

and we thence have

$$\alpha^3 X + \beta^3 Y + \gamma^3 Z = 3\alpha\beta\gamma X^{\frac{1}{3}} Y^{\frac{1}{3}} Z^{\frac{1}{3}}.$$

The left-hand side is

$$\begin{aligned} &\alpha^3\ (A + 3Bx + 3Cx^2 + Dx^3) \\ &+ \beta^3\ (A + 3By + 3Cy^2 + Dy^3) \\ &+ \gamma^3\ (A + 3Bz + 3Cz^2 + Dz^3)\,; \end{aligned}$$

or assuming

$$\alpha',\ \beta',\ \gamma' = x\,(y - z),\ y\,(z - x),\ z\,(x - y) \ \text{(so that } \alpha' + \beta' + \gamma' = 0\text{)},$$

this is

$$= A\,(\alpha^3 + \beta^3 + \gamma^3) + 3B\,(\alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma') + 3C\,(\alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2) + D\,(\alpha'^3 + \beta'^3 + \gamma'^3).$$

But taking λ arbitrary, and

$$a,\ b,\ c = \alpha + \lambda\alpha',\quad \beta + \lambda\beta',\quad \gamma + \lambda\gamma',$$

then

$$a + b + c = 0,$$

whence

$$a^3 + b^3 + c^3 = 3abc\,;$$

or substituting for a, b, c their values, and comparing the coefficients of the several powers of λ,

$$\begin{aligned}
&\alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma,\\
&\alpha^2\alpha' + \beta^2\beta' + \gamma^2\gamma' = \alpha'\beta\gamma + \beta'\gamma\alpha + \gamma'\alpha\beta = \alpha\beta\gamma\,(x + y + z),\\
&\alpha\alpha'^2 + \beta\beta'^2 + \gamma\gamma'^2 = \alpha\beta'\gamma' + \beta\gamma'\alpha' + \gamma\alpha'\beta' = \alpha\beta\gamma\,(yz + zx + xy),\\
&\alpha'^3 + \beta'^3 + \gamma'^3 = 3\alpha'\beta'\gamma' = 3\alpha\beta\gamma\,.\,xyz.
\end{aligned}$$

Hence we have

$$\alpha^3 X + \beta^3 Y + \gamma^3 Z = 3\alpha\beta\gamma\,\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\},$$

or the integral equation is

$$\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\} = X^{\frac{1}{3}}Y^{\frac{1}{3}}Z^{\frac{1}{3}},$$

that is,

$$\{A + B(x+y+z) + C(yz+zx+xy) + Dxyz\}^3 = XYZ,$$

the elegant result given by Capt. MacMahon at the beginning of his paper.

The author in a letter to me, dated Jan. 13, 1883, remarks that the particular integral of the equation in question

$$X^{-\frac{2}{3}}\,dx + Y^{-\frac{2}{3}}\,dy + Z^{-\frac{2}{3}}\,dz = 0,$$

is expressible as a determinant in a rational form as follows. Writing it $XYZ = P^3$, where

$$P = A + B(x+y+z) + C(yz+zx+xy) + Dxyz,$$

then the form is

$$\nabla, = \begin{vmatrix} 1, & \left(\tfrac{1}{3}P\dfrac{dX}{dx} - X\dfrac{dP}{dx}\right)^3, & X \\ 1, & \left(\tfrac{1}{3}P\dfrac{dY}{dy} - Y\dfrac{dP}{dy}\right)^3, & Y \\ 1, & \left(\tfrac{1}{3}P\dfrac{dZ}{dz} - Z\dfrac{dP}{dz}\right)^3, & Z \end{vmatrix} = 0,$$

for, as shown by Captain MacMahon in his paper, each of the three terms such as

$$Z\left(\tfrac{1}{3}P\frac{dY}{dy} - Y\frac{dP}{dy}\right)^3 - Y\left(\tfrac{1}{3}P\frac{dZ}{dz} - Z\frac{dP}{dz}\right)^3,$$

which compose the determinant, is divisible by $XYZ - P^3$.

It may be added that we have identically

$$\tfrac{1}{3}P\frac{dX}{dx} - X\frac{dP}{dx} = (AC - B^2)(2x - y - z) + (AD - BC)(x^2 - yz) + (BD - C^2)(x^2y + x^2z - 2xyz),$$

and of course like values for the other two expressions in the determinant.