801.

ON SEMINVARIANTS.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. XIX. (1883), pp. 131—138.]

THE present paper is a somewhat fragmentary one, but it contains some results which seem to me to be worth putting on record.

I consider here not any binary quantic in particular, but the whole series $(a, b, c \between x, y)^2$, $(a, b, c, d \between x, y)^3$, &c.; or in a somewhat different point of view, I consider the indefinite series of coefficients $(a, b, c, d, e, \ldots)$; here, instead of covariants and invariants, we have only seminvariants; viz. a seminvariant is a function reduced to zero by the operator

$$\Delta = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots;$$

for instance, seminvariants are

$$a,\quad ac - b^2,\quad a^2d - 3abc + 2b^3,\quad a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$
$$ae - 4bd + 3c^2,\quad ace - ad^2 - b^2e + 2bcd - c^3, \text{ \&c.}$$

A seminvariant is of a certain degree θ in the coefficients, and of a certain weight w (viz. the coefficients $a, b, c, d, \ldots$ are reckoned as being of the weights $0, 1, 2, 3, \ldots$ respectively); it is, moreover, of a certain rank ρ; viz. according as the highest letter therein is $a, c, d, e, \ldots$ (it is never b), the rank is taken to be $0, 2, 3, 4, \ldots$, and we have $w =$ or $< \frac{1}{2}\rho\theta$. The seminvariant may be regarded as belonging to a quantic $(a, \ldots \between x, y)^n$, the order of which, n, is equal to or greater than ρ; viz. in regard to such quantic the seminvariant, say A, is the leading coefficient of a covariant

$$(A, B, \ldots, K \between x, y)^\mu,$$

where the weights of the successive coefficients are $w, w+1, \ldots$ up to $n\theta - w$; hence number of terms less unity, that is, μ, is $= n\theta - 2w$; the least value of μ is thus $= \rho\theta - 2w$, which is either zero, or positive; in the former case, $w = \frac{1}{2}\rho\theta$, the seminvariant is an invariant of the quantic $(a, \ldots \between x, y)^\rho$, the order of which is equal to the rank of the seminvariant; but if $w < \frac{1}{2}\rho\theta$, then it is the leading coefficient of a covariant $(A, B, \ldots, K \between x, y)^{\rho\theta - 2w}$ of the same quantic $(a, \ldots \between x, y)^\rho$; and in every case, taking $n > \rho$, the seminvariant is the leading coefficient A of a covariant

$$(A, B, \ldots, K \between x, y)^{n\theta - 2w}$$

of a quantic $(a, \ldots \between x, y)^n$.

Take A as belonging to the quantic $(a, \ldots \between x, y)^n$; corresponding to such quantic, we have an operator Λ of the same rank n, viz.

$$\begin{aligned} \Lambda &= 2b\partial_a + c\partial_b && \text{for } n = 2, \\ &= 3b\partial_a + 2c\partial_b + d\partial_c && \quad ,, \quad 3, \\ &= 4b\partial_a + 3c\partial_b + 2d\partial_c + e\partial_d && \quad ,, \quad 4, \\ &\vdots && \qquad \vdots \end{aligned}$$

Operating with Λ on A, we have a series of terms

$$A, \Lambda A, \Lambda^2 A, \ldots, \Lambda^{n\theta - 2w} A,$$

but the next term $\Lambda^{n\theta-2w+1}A$, and of course every succeeding term, is $=0$, and this being so, the coefficients of the covariant $(A, B, \ldots, K \between x, y)^{n\theta-2w}$ are

$$(1, \tfrac{1}{1}\Lambda, \tfrac{1}{1 \cdot 2}\Lambda^2, \ldots) A,$$

or what is the same thing, each coefficient is obtained from the next preceding one by the formulæ

$$B = \tfrac{1}{1}\Lambda A, \quad C = \tfrac{1}{2}\Lambda B, \quad D = \tfrac{1}{3}\Lambda C, \ldots.$$

The coefficients A and K, B and $J, \ldots$ are derived one from the other by reversal of the order of the coefficients of $(a, b, \ldots \between x, y)^n$, with or without a change of sign, and thus it is only necessary to calculate up to the middle coefficient, or pair of coefficients; and we obtain, moreover, a verification.

Calculating in this manner the covariant

$$(A, B, \ldots, K)^{\rho\theta - 2w},$$

which belongs to the quantic $(a, \ldots \between x, y)^\rho$, if we herein change $a, b, c, \ldots$ into $ax + by$, $bx + cy$, $cx + dy, \ldots$ we obtain the covariant belonging to the quantic $(a, \ldots \between x, y)^{\rho+1}$; and in this covariant making the like change, or what is the same thing, in the first-mentioned covariant changing $a, b, c, \ldots$ into $(a, b, c \between x, y)^2$, $(b, c, d \between x, y)^2$, $(c, d, e \between x, y)^2, \ldots$ we have the covariant belonging to $(a, \ldots \between x, y)^{\rho+2}$; and in like manner we obtain the covariant belonging to the quantic $(a, \ldots \between x, y)^n$ of any given order n.

In particular, if $w = \frac{1}{2}\rho\theta$, that is, if the given seminvariant be an invariant of $(a, \ldots \between x, y)^\rho$, then we obtain the series of covariants directly from A by therein changing $a, b, c, \ldots$ into $ax + by$, $bx + cy$, $cx + dy, \ldots$ and in the result making the like change; or what is the same thing, in A changing $a, b, c, \ldots$ into $(a, b, c \between x, y)^2$, $(b, c, d \between x, y)^2$, $(c, d, e \between x, y)^2, \ldots$: and so on until we obtain the covariant for the quantic $(a, \ldots \between x, y)^n$ of the given order n.

A seminvariant which cannot be expressed as a rational and integral function of lower seminvariants is said to be irreducible. The theory is distinct from that of the irreducible covariants of a quantic of a given order; for instance, as regards the cubic $(a, b, c, d \between x, y)^3$, we have the irreducible covariant (invariant)

$$a^2d^2 + 4ac^3 + 4b^3d - 6abcd - 3b^2c^2,$$

but this is not an irreducible seminvariant; it is

$$\begin{aligned} &= (ac - b^2)(ae - 4bd + 3c^2) \\ &\quad - a \,.\, (ace - ad^2 - b^2e - c^3 + 2bcd), \end{aligned}$$

or, what is the same thing, there is not for the quartic $(a, b, c, d, e \between x, y)^4$, or for the higher quantics, any *irreducible* covariant having this for the leading coefficient.

We may consider the question to determine the number of asyzygetic seminvariants of a given degree and weight. For instance, taking the weights up to 12, so that the series of letters extends as far as m, then for the degrees 1, 2, 3 we have as follows:

$W=$	0	1	2	3	4	5	6	7	8	9	10	11	12
Deg. 1	a	b	c	d	e	f	g	h	i	j	k	l	m
Nos.	1	1	1	1	1	1	1	1	1	1	1	1	1
Diff.	0	0	0	0	0	0	0	0	0	0	0	0	0
Deg. 2	a^2	ab	ac b^2	ad bc	ae bd c^2	af be cd	ag bf ce d^2	ah bg cf de	ai bh cg df e^2	aj bi ch dg ef	ak bj ci dh eg f^2	al bk cj di eh fg	am bl ck dj ei fh g^2
Nos.	1	1	2	2	3	3	4	4	5	5	6	6	7
Diff.	1	0	1	0	1	0	1	0	1	0	1	0	1
Deg. 3	a^3	a^2b	a^2c ab^2	a^2d abc b^3	a^2e abd ac^2 b^2c	a^2f abe acd b^2d bc^2	a^2g abf ace ad^2 b^2e bcd c^3	a^2h abg acf ade b^2f bce bd^2 c^2d	a^2i abh acg adf ae^2 b^2g bcf bde c^2e cd^2	a^2j abi ach adg aef b^2h bcg bdf be^2 c^2f cde d^3	a^2k abj aci adh aeg af^2 b^2i bch bdg bef c^2g cdf ce^2 d^2e	a^2l abk acj adi aeh afg b^2j bci bdh beg bf^2 c^2h cdg cef d^2f de^2	a^2m abl ack adj aei afh ag^2 b^2k bcj bdi beh bfg c^2i cdh ceg cf^2 d^2g def e^3
Nos.	1	1	2	3	4	5	7	8	10	12	14	16	19
Diff.	1	0	1	1	1	2	1	2	2	2	2	2	3

For the degree 1, the line of differences shows that the only seminvariant is ($W=0$), the seminvariant a.

For the degree 2, the line of differences 1, 0, 1, 0, ..., shows that the number of seminvariants is $=1$ for each even degree, $=0$ for each odd degree; thus for the weight 0 there is a seminvariant $=a^2$, which of course is not irreducible; while for each of the other even weights we have a single irreducible seminvariant; as is well known, the forms are

$W=$	2	4	6	8	10	12
	$ac+1$	$ae+1$	$ag+1$	$ai+1$	$ak+1$	$am+1$
	b^2-1	$bd-4$	$bf-6$	$bh-8$	$bj-10$	$bl-12$
		c^2+3	$ce+15$	$cg+28$	$ci+45$	$ck+66$
			d^2-10	$df-56$	$dh-120$	$dj-220$
				e^2+35	$eg+210$	$ei+495$
					f^2-126	$fh-792$
						g^2+462

For degree 3, the line of differences shows that for

W	=	0	1	2	3	4	5	6	7	8	9	10	11	12
Nos. are	=	1	0	1	1	1	1	2	1	2	2	2	2	3 ;

but inasmuch as for each even weight there is a quadric seminvariant, which multiplied by a gives a cubic seminvariant, to obtain the number of irreducible cubic seminvariants we subtract

1	0	1	0	1	0	1	0	1	0	1	0	1
0	0	0	1	0	1	1	1	1	2	1	2	2 ,

or the numbers of irreducible cubic seminvariants are as in the line last written down.

There is a convenience however in giving, for each even weight, as well the rejected reducible covariant; and the entire series of results is found to be

$W=$ 0	2	3	4	5	6		7	8	
a^3 + 1	a^2c + 1	a^2d + 1	a^2e + 1	a^2f + 1	a^2g + 1		a^2h + 1	a^2i + 1	
	ab^2 − 1	abc − 3	abd − 4	abe − 5	abf − 6		abg − 7	abh − 8	
		b^3 + 2	ac^2 + 3	acd + 2	ace + 15	+ 1	acf + 9	acg + 28	+ 1
				b^2d + 8	ad^2 − 10	− 1	ade − 5	adf − 56	− 3
				bc^2 − 6	b^2e	− 1	b^2f + 12	ae^2 + 35	+ 2
					bcd	+ 2	bce − 30	b^2g	− 1
					c^3	− 1	bd^2 + 20	bcf	+ 3
							c^2d	bde	− 1
								c^2e	− 3
								cd^2	+ 2

9		10		11		12		
a^2j + 1		a^2k + 1		a^2l + 1		a^2m + 1		
abi − 9		abj − 10		abk − 11		abl − 12		
ach	+ 2	aci + 45	+ 1	acj + 35	+ 2	ack + 66	+ 3	
adg + 42	− 7	adh − 120	− 4	adi − 75	− 9	adj − 220	− 15	
aef − 36	+ 5	aeg + 210	+ 8	aeh + 90	+ 14	aei + 495	+ 40	+ 1
b^2h + 36	− 2	af^2 − 126	− 5	afg − 42	− 7	afh − 792	− 70	− 4
bcg − 126	+ 7	b^2i	− 1	b^2j + 20	− 2	ag^2 + 462	+ 42	+ 3
bdf − 108	+ 22	bch	+ 4	bci − 90	+ 9	b^2k	− 3	
be^2 + 180	− 25	bdg	− 4	bdh + 240	+ 16	bcj	+ 15	
c^2f + 270	− 27	bef	+ 2	beg − 420	− 63	bdi	− 25	− 4
cde − 450	+ 45	c^2g	− 4	bf^2 + 252	+ 42	beh	+ 30	+ 12
d^3 + 200	− 20	cdf	+ 8	c^2h	− 30	bfg	− 14	− 8
		ce^2	− 5	cdg	+ 70	c^2i	− 15	+ 3
		d^2e		cef	− 21	cdh	+ 40	− 8
				d^2f	− 56	ceg	− 70	− 22
				de^2	+ 35	cf^2	+ 42	+ 24
						d^2g		+ 24
						def		− 36
						e^3		+ 15

The canonical form given for the quintic in my Tenth Memoir on Quantics [693] belongs to a series, viz. writing now the small roman letters (instead of the italic letters) for the series of coefficients, and using the italic letters a, c, d, e, f, ... to denote seminvariants, they are as follows:

0 $a = \mathrm{a}$,

2 $c = \mathrm{ac} - \mathrm{b}^2$,

3 $d = \mathrm{a}^2\mathrm{d} - 3\mathrm{abc} + 2\mathrm{b}^3$ ($= f$ in the tenth memoir),

4 $e = \mathrm{ae} - 4\mathrm{bd} + 3\mathrm{c}^2$ ($= b$ in the tenth memoir),

5 $f = \mathrm{a}^2\mathrm{f} - 5\mathrm{abe} + 2\mathrm{acd} + 8\mathrm{b}^2\mathrm{d} - 6\mathrm{bc}^2$,

6 $g = \mathrm{ag} - 6\mathrm{bf} + 15\mathrm{ce} - 10\mathrm{d}^2$,

7 $h = \mathrm{a}^2\mathrm{h} - 7\mathrm{abg} + 9\mathrm{acf} - 5\mathrm{ade} + 12\mathrm{b}^2\mathrm{f} - 30\mathrm{bce} + 20\mathrm{bd}^2$,

8 $i = \mathrm{ai} - 8\mathrm{bh} + 28\mathrm{cg} - 56\mathrm{df} + 35\mathrm{e}^2$,

&c.

Writing also (instead of d in the tenth memoir)

$$\epsilon = \mathrm{ace} - \mathrm{ad}^2 - \mathrm{b}^2\mathrm{e} + 2\mathrm{bcd} - \mathrm{c}^3,$$

so that the equation $a^3d - a^2bc + 4c^3 - f^2 = 0$ of the tenth memoir, is in the present notation $a^3\epsilon - a^2ec + 4c^3 - d^2 = 0$, then the series of canonical forms is

Quadric $(1,\ 0,\ c \between x,\ y)^2$,

Cubic $(1,\ 0,\ c,\ d \between x,\ y)^3$,

Quartic $(1,\ 0,\ c,\ d,\ a^2e - 3c^2 \between x,\ y)^4$,

Quintic $(1,\ 0,\ c,\ d,\ a^2e - 3c^2,\ a^2f - 2cd \between x,\ y)^5$,

&c.

the series of coefficients being

1,	0,	c,	d,	$a^2e + 1$,	$a^2f + 1$,	$a^4g + 1$,	$a^4h + 1$,	$a^6i + 1$,
				$c^2 - 3$	$cd - 2$	$a^2ce - 15$	$a^2cf - 9$	$a^4cg - 28$
						$c^3 + 45$	$a^2de + 5$	$a^4e^2 - 35$
						$d^2 + 10$	$c^2d + 3$	$a^2c^2e + 630$
								$a^2df + 56$
								$c^4 - 1575$
								$cd^2 - 392$

these values being, in fact, the expressions in terms of the seminvariants a, c, d, &c. of

1,	0,	ac,	$\mathrm{a}^2\mathrm{d}$,	$\mathrm{a}^3\mathrm{e}$,	$\mathrm{a}^4\mathrm{f}$,	$\mathrm{a}^5\mathrm{g}$,	$\mathrm{a}^6\mathrm{h}$,	$\mathrm{a}^7\mathrm{i}$
		$-\mathrm{b}^2$,	$-3\mathrm{abc}$,	$-4\mathrm{a}^2\mathrm{bd}$,	$-5\mathrm{a}^3\mathrm{be}$,	$-6\mathrm{a}^4\mathrm{bf}$,	$-7\mathrm{a}^5\mathrm{bg}$,	$-8\mathrm{a}^6\mathrm{bh}$
			$+2\mathrm{b}^3$,	$+6\mathrm{ab}^2\mathrm{c}$,	$+10\mathrm{a}^2\mathrm{b}^2\mathrm{d}$,	$+15\mathrm{a}^3\mathrm{b}^2\mathrm{e}$,	$+21\mathrm{a}^4\mathrm{b}^2\mathrm{f}$,	$+28\mathrm{a}^5\mathrm{b}^2\mathrm{g}$
				$-3\mathrm{b}^4$,	$-10\mathrm{ab}^3\mathrm{c}$,	$-20\mathrm{a}^2\mathrm{b}^3\mathrm{d}$,	$-35\mathrm{a}^3\mathrm{b}^3\mathrm{e}$,	$-56\mathrm{a}^4\mathrm{b}^3\mathrm{f}$
					$+4\mathrm{b}^5$,	$+15\mathrm{ab}^4\mathrm{c}$,	$+35\mathrm{a}^2\mathrm{b}^4\mathrm{d}$,	$+70\mathrm{a}^3\mathrm{b}^4\mathrm{e}$
						$-5\mathrm{b}^6$,	$-21\mathrm{ab}^5\mathrm{c}$,	$-56\mathrm{a}^2\mathrm{b}^5\mathrm{d}$
							$+6\mathrm{b}^7$,	$+28\mathrm{ab}^6\mathrm{c}$
								$-7\mathrm{b}^8$

4—2

I annex verifications of the foregoing values:

	$a^2e=$	$-3c^2$	
a^3e	1		+1
a^2bd	−4		−4
a^2c^2	+3	−3	0
ab^2c		+6	+6
b^4		−3	−3

	$a^2f=$	$-2cd$	
a^4f	+1		+1
a^3be	−5		−5
a^3cd	+2	−2	0
a^2b^2d	+8	+2	+10
a^2bc^2	−6	+6	0
ab^3c		−10	−10
b^5		+4	+4

	$a^4g=$	$-15a^2ce$	$+45c^3$	$+10d^2$	
a^5g	+1				1
a^4bf	−6				−6
a^4ce	+15	−15			0
a^4d^2	−10			+10	0
a^3b^2e		+15			+15
a^3bcd		+60		−60	0
a^3c^3		−45	+45		0
a^2b^3d		−60		+40	−20
$a^2b^2c^2$		+45	−135	+90	0
ab^4c			+135	−120	+15
b^6			−45	+40	−5

	$a^4h=$	$-9a^2cf$	$+5a^2de$	$+3c^2d$	
a^6h	+1				+1
a^5bg	−7				−7
a^5cf	+9	−9			0
a^5de	−5		+5		0
a^4b^2f	+12	+9			+21
a^4bce	−30	+45	−15		0
a^4bd^2	+20		−20		0
a^4c^2d		−18	+15	+3	0
a^3b^3e		−45	+10		−35
a^3b^2cd		−54	+60	−6	0
a^3bc^3		+54	−45	−9	0
a^2b^4d		+72	−40	+3	+35
$a^2b^3c^2$		−54	+30	+24	0
ab^5c				−21	−21
b^7				+6	+6

	$a^6i=$	$-28a^4cg$	$-35a^4e^2$	$+56a^2df$	$+630a^2c^2e^2$	$-1575c^4$	$-392c^2d$	
a^7i	+ 1							+ 1
a^6bh	− 8							− 8
cg	+ 28	− 28						0
df	− 56				+ 56			0
e^2	+ 35		− 35					0
a^5b^2g		+ 28						+ 28
bcf		+ 168			− 168			0
bde			+ 280		− 280			0
c^2e		− 420	− 210	+ 630				0
cd^2		+ 280			+ 112		− 392	0
a^4b^3f		− 168			+ 112			− 56
b^2ce		+ 420		− 1260	+ 840			0
b^2d^2		− 280	− 560		+ 448		+ 392	0
bc^2d			+ 840	− 2520	− 672		+ 2352	0
c^4			− 315	+ 1890		− 1575		0
a^3b^4e				+ 630	− 560			+ 70
b^3cd				+ 5040	− 1120		− 3920	0
b^2c^3				− 3780	+ 1008	+ 6300	− 3528	0
a^2b^5d				− 2520	+ 896		+ 1568	− 56
b^4c^2				+ 1890	− 672	− 9450	+ 8232	0
$a\,b^6d$						+ 6300	− 6272	+ 28
a^0b^8						− 1575	+ 1568	− 7

It would be interesting to obtain the general law for the expressions of the canonical coefficients in terms of the seminvariants.