799.

ON CURVILINEAR COORDINATES.

THE present memoir is based upon Mr Warren's "Exercises in Curvilinear and Normal Coordinates," *Camb. Phil. Trans.* t. XII. (1877), pp. 455—502, and has for a principal object the establishment of the six differential equations of the second order corresponding to his six equations for normal coordinates: but the notation is different; the results are more general, inasmuch as I use throughout general curvilinear coordinates instead of his normal coordinates; and as regards my six equations for general curvilinear coordinates, the terms containing differential coefficients of the first order are presented under a different form.

If the position of a point in space is determined by the rectangular coordinates x, y, z; then p, q, r being each of them a given function of x, y, z, we have conversely x, y, z, each of them a given function of p, q, r, which are thus in effect coordinates serving to determine the position of the point, and are called curvilinear coordinates.

But it is not in the first instance necessary to regard x, y, z as rectangular coordinates, or even as Cartesian coordinates at all; we are simply concerned with the two sets of variables x, y, z and p, q, r, each variable of the one set being a given function of the variables of the other set; and, in particular, the x, y, z are regarded as being each of them a given function of the p, q, r.

Except as regards the symbols ξ, η, ζ presently mentioned, the suffixes 1, 2, 3 refer to the variables p, q, r respectively, and are used to denote differentiations in regard to these variables, viz.

$$x_1, \ x_2, \ x_3, \ x_{11}, \ x_{12}, \ x_{22}, \ \ldots$$

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are written to denote

$$\frac{dx}{dp}, \quad \frac{dx}{dq}, \quad \frac{dx}{dr}, \quad \frac{d^2x}{dp^2}, \quad \frac{d^2x}{dp\,dq}, \quad \frac{d^2x}{dq^2}, \quad \&c.,$$

and so in other cases; in particular,

$$x_1, \ x_2, \ x_3 \text{ denote } \frac{dx}{dp}, \ \frac{dx}{dq}, \ \frac{dx}{dr},$$

$$y_1, \ y_2, \ y_3 \quad \text{,,} \quad \frac{dy}{dp}, \ \frac{dy}{dq}, \ \frac{dy}{dr},$$

$$z_1, \ z_2, \ z_3 \quad \text{,,} \quad \frac{dz}{dp}, \ \frac{dz}{dq}, \ \frac{dz}{dr}.$$

The minors formed with these differential coefficients are denoted by suffixed letters ξ, η, ζ, thus

$$\xi_1, \ \xi_2, \ \xi_3 \text{ denote } y_2z_3 - y_3z_2, \quad y_3z_1 - y_1z_3, \quad y_1z_2 - y_2z_1,$$

$$\eta_1, \ \eta_2, \ \eta_3 \quad \text{,,} \quad z_2x_3 - z_3x_2, \quad z_3x_1 - z_1x_3, \quad z_1x_2 - z_2x_1,$$

$$\zeta_1, \ \zeta_2, \ \zeta_3 \quad \text{,,} \quad x_2y_3 - x_3y_2, \quad x_3y_1 - x_1y_3, \quad x_1y_2 - x_2y_1,$$

so that, as regards these letters ξ, η, ζ, the suffixes do *not* denote differentiations.

The determinant $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}$ is put $= L$:

and the symbols (a, b, c, f, g, h), (A, B, C, F, G, H) are defined as follows:

$$
\begin{aligned}
a &= x_1^2 + y_1^2 + z_1^2, & A &= \xi_1^2 + \eta_1^2 + \zeta_1^2, \\
b &= x_2^2 + y_2^2 + z_2^2, & B &= \xi_2^2 + \eta_2^2 + \zeta_2^2, \\
c &= x_3^2 + y_3^2 + z_3^2, & C &= \xi_3^2 + \eta_3^2 + \zeta_3^2, \\
f &= x_2x_3 + y_2y_3 + z_2z_3, & F &= \xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3, \\
g &= x_3x_1 + y_3y_1 + z_3z_1, & G &= \xi_3\xi_1 + \eta_3\eta_1 + \zeta_3\zeta_1, \\
h &= x_1x_2 + y_1y_2 + z_1z_2, & H &= \xi_1\xi_2 + \eta_1\eta_2 + \zeta_1\zeta_2.
\end{aligned}
$$

We have then, further,

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = L, \qquad \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = L^2,$$

$$\begin{vmatrix} \xi_1, & \eta_1, & \zeta_1 \\ \xi_2, & \eta_2, & \zeta_2 \\ \xi_3, & \eta_3, & \zeta_3 \end{vmatrix} = L^2, \qquad \begin{vmatrix} A, & H, & G \\ H, & B, & F \\ G, & F, & C \end{vmatrix} = \frac{1}{L^2},$$

$$(A, \ B, \ C, \ F, \ G, \ H) = (bc - f^2, \ ca - g^2, \ ab - h^2, \ gh - af, \ hf - bg, \ fg - ch),$$

$$L^2(a, \ b, \ c, \ f, \ g, \ h) = (BC - F^2, \ CA - G^2, \ AB - H^2, \ GH - AF, \ HF - BG, \ FG - CH),$$

which equations are at once proved, and are fundamental ones in the theory.

It is convenient to add that we have

$$
\begin{pmatrix}
\dfrac{dp}{dx}, & \dfrac{dp}{dy}, & \dfrac{dp}{dz} \\[2mm]
\dfrac{dq}{dx}, & \dfrac{dq}{dy}, & \dfrac{dq}{dz} \\[2mm]
\dfrac{dr}{dx}, & \dfrac{dr}{dy}, & \dfrac{dr}{dz}
\end{pmatrix}
= \frac{1}{L}
\begin{pmatrix}
\xi_1, & \eta_1, & \zeta_1 \\
\xi_2, & \eta_2, & \zeta_2 \\
\xi_3, & \eta_3, & \zeta_3
\end{pmatrix},
$$

that is, $\dfrac{dp}{dx} = \dfrac{1}{L}\,\xi_1$, &c.; and, further,

$$
dx^2 + dy^2 + dz^2 = \quad (a,\ b,\ c,\ f,\ g,\ h\ \mathbb{\chi} dp,\ dq,\ dr)^2,
$$

$$
dp^2 + dq^2 + dr^2 = \frac{1}{L^2}(A,\ B,\ C,\ F,\ G,\ H\ \mathbb{\chi} dx,\ dy,\ dz)^2.
$$

Differentiating the values of a, b, c, f, g, h, we have $\frac{1}{2}a_1 = x_1 x_{11} + y_1 y_{11} + z_1 z_{11}$, which may be written in the abbreviated form $\frac{1}{2}a_1 = 1.11$; similarly

$$
f_1 = x_2 x_{13} + y_2 y_{13} + z_2 z_{13} + x_3 x_{12} + y_3 y_{12} + z_3 z_{12},
$$

which in like manner may be written $f_1 = 2.13 + 3.12$, and so in other cases; observe that, in the duad part of any symbol, the order of the numbers is immaterial, $2.13 = 2.31$. The whole system of equations is

$$
\begin{array}{llll}
\tfrac{1}{2}a_1 = 1.11 & , & \tfrac{1}{2}a_2 = 1.12 & , \quad \tfrac{1}{2}a_3 = 1.13 \qquad , \\[1mm]
\tfrac{1}{2}b_1 = 2.12 & , & \tfrac{1}{2}b_2 = 2.22 & , \quad \tfrac{1}{2}b_3 = 2.23 \qquad , \\[1mm]
\tfrac{1}{2}c_1 = 3.13 & , & \tfrac{1}{2}c_2 = 3.23 & , \quad \tfrac{1}{2}c_3 = 3.33 \qquad , \\[2mm]
f_1 = 2.13 + 3.12, & f_2 = 2.23 + 3.22, & f_3 = 2.33 + 3.23, \\[1mm]
g_1 = 3.11 + 1.13, & g_2 = 3.12 + 1.23, & g_3 = 3.13 + 1.33, \\[1mm]
h_1 = 1.12 + 2.11, & h_2 = 1.22 + 2.12, & h_3 = 1.23 + 2.13.
\end{array}
$$

These may also be written

$$
\begin{array}{llll}
1.11 = \tfrac{1}{2}a_1 & , & 1.22 = h_2 - \tfrac{1}{2}b_1 & , \quad 1.33 = g_3 - \tfrac{1}{2}c_1 \qquad , \\[1mm]
2.11 = h_1 - \tfrac{1}{2}a_2 & , & 2.22 = \tfrac{1}{2}b_2 & , \quad 2.33 = f_3 - \tfrac{1}{2}c_2 \qquad , \\[1mm]
3.11 = g_1 - \tfrac{1}{2}a_3 & , & 3.22 = f_2 - \tfrac{1}{2}b_3 & , \quad 3.33 = \tfrac{1}{2}c_3 \qquad , \\[1mm]
1.23 = \tfrac{1}{2}(-f_1 + g_2 + h_3), & 1.31 = \tfrac{1}{2}a_3 & , & 1.12 = \tfrac{1}{2}a_2 \qquad , \\[1mm]
2.23 = \tfrac{1}{2}b_3 & , & 2.31 = \tfrac{1}{2}(f_1 - g_2 + h_3), & 2.12 = \tfrac{1}{2}b_1 \qquad , \\[1mm]
3.23 = \tfrac{1}{2}c_2 & , & 3.31 = \tfrac{1}{2}c_1 & , \quad 3.12 = \tfrac{1}{2}(f_1 + g_2 - h_3).
\end{array}
$$

It is to be observed that we can, from each system of three equations, express a set of second differential coefficients of the x, y, z in terms of the first differential

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coefficients of the a, b, c, f, g, h; thus the three equations containing 11, written at length, are

$$x_1 . x_{11} + y_1 . y_{11} + z_1 . z_{11} = \tfrac{1}{2}a_1,$$
$$x_2 \ " \ + y_2 \ " \ + z_2 \ " \ = h_1 - \tfrac{1}{2}a_2,$$
$$x_3 \ " \ + y_3 \ " \ + z_3 \ " \ = g_1 - \tfrac{1}{2}a_3,$$

three linear equations for the determination of x_{11}, y_{11}, z_{11}; hence for x_{11} we have

$$x_{11} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \tfrac{1}{2}a_1 (y_2 z_3 - y_3 z_2) + (h_1 - \tfrac{1}{2}a_2)(y_3 z_1 - y_1 z_3) + (g_1 - \tfrac{1}{2}a_3)(y_1 z_2 - y_2 z_1),$$

or, what is the same thing,

$$Lx_{11} = \tfrac{1}{2}a_1 . \xi_1 + (h_1 - \tfrac{1}{2}a_2) \xi_2 + (g_1 - \tfrac{1}{2}a_3) \xi_3,$$

and so for y_{11}, z_{11}; or if (as in the sequel) we desire the value of a linear function $\alpha x_{11} + \beta y_{11} + \gamma z_{11}$, calling this for a moment $\square$, we join to the foregoing a new equation

$$\alpha x_{11} + \beta y_{11} + \gamma z_{11} = \square,$$

and then, eliminating the three quantities, we have

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}a_1 \\ x_2, & y_2, & z_2, & h_1 - \tfrac{1}{2}a_2 \\ x_3, & y_3, & z_3, & g_1 - \tfrac{1}{2}a_3 \end{vmatrix} = 0,$$

giving $L\square$ as a linear function of $\tfrac{1}{2}a_1$, $h_1 - \tfrac{1}{2}a_2$, $g_1 - \tfrac{1}{2}a_3$.

We can in like manner form the expressions for the second differential coefficients of the a, b, c, f, g, h: these will of course contain third differential coefficients of the x, y, z.

Writing down only what is wanted, we have

$$\tfrac{1}{2}a_{22} = 12 . 12 + 1 . 122,$$
$$\tfrac{1}{2}b_{11} = 12 . 12 + 2 . 112,$$
$$h_{12} = 12 . 12 + 11 . 22 + 1 . 122 + 2 . 112,$$

where of course $12 . 12$ denotes $x^2_{12} + y^2_{12} + z^2_{12}$, $1 . 122$ denotes $x_1 . x_{122} + y_1 . y_{122} + z_1 . z_{122}$, and so in other cases: it follows that

$$\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12}) = 12 . 12 - 11 . 22,$$

so that the third differential coefficients of x, y, z, which enter into the expression of $a_{22} + b_{11} - 2h_{12}$ destroy each other, and this combination contains really only second differential coefficients of x, y, z.

Similarly

$$f_{31} = \quad 31.23 + 33.12 + \quad 3.123 \qquad\qquad + 2.133,$$
$$g_{23} = \quad 31.23 + 33.12 + \quad 3.123 + 1.233$$
$$c_{12} = 2.31.23 \qquad\qquad + 2.3..123$$
$$h_{33} = 2.31.23 \qquad\qquad\qquad\qquad + 1.233 + 2.133,$$

and thence

$$f_{31} + g_{23} - c_{12} - h_{33} = -2(31.23 - 33.12),$$

so that here again we have a combination containing only second differential coefficients of x, y, z.

There are thus, in all, the six combinations

$$b_{33} + c_{22} - 2f_{23} \dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{A}),$$
$$c_{11} + a_{33} - 2g_{31} \dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{B}),$$
$$a_{22} + b_{11} - 2h_{12} \dots\dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{C}),$$
$$g_{12} + h_{31} - a_{23} - f_{11} \dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{F}),$$
$$h_{23} + f_{12} - b_{31} - g_{22} \dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{G}),$$
$$f_{31} + g_{23} - c_{12} - h_{33} \dots\dots\dots\dots\dots\dots\dots\dots(\mathfrak{H}),$$

each really containing only second differential coefficients of x, y, z; and we thus understand how each of these combinations may be expressible in terms of the first differential coefficients of (a, b, c, f, g, h). We have, in fact, for thus expressing these combinations the six equations called $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$, $(\mathfrak{F})$, $(\mathfrak{G})$, $(\mathfrak{H})$ about to be obtained, and which are the generalisations of Warren's six equations for normal coordinates.

I consider the several determinants of the form

$$\begin{vmatrix} x_{11}, & y_{11}, & z_{11} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

in all 18, since the suffixes for the top row may be 11, 22, 33, 23, 31, 12, and those for the second and third rows 2, 3; 3, 1; or 1, 2. Each determinant is a linear function of second differential coefficients of x, y, z (thus the determinant written down is $= \xi_3 x_{11} + \eta_3 y_{11} + \zeta_3 z_{11}$), and as such it can, by what precedes, be expressed by means of the first differential coefficients of the a, b, c, f, g, h. Thus, if the determinant above written down be called $\square$, writing ξ_3, η_3, ζ_3 for α, β, γ, we have

$$\begin{vmatrix} \xi_3, & \eta_3, & \zeta_3, & \square \\ x_1, & y_1, & z_1, & \tfrac{1}{2}a_1 \\ x_2, & y_2, & z_2, & h_1 - \tfrac{1}{2}a_2 \\ x_3, & y_3, & z_3, & g_1 - \tfrac{1}{2}a_3 \end{vmatrix} = 0,$$

that is,

$$L.\square = - \begin{vmatrix} & \xi_3, & \eta_3, & \zeta_3 \\ \tfrac{1}{2}a_1, & x_1, & y_1, & z_1 \\ h_1 - \tfrac{1}{2}a_2, & x_2, & y_2, & z_2 \\ g_1 - \tfrac{1}{2}a_3, & x_3, & y_3, & z_3 \end{vmatrix},$$

where the right-hand side is

$$= \tfrac{1}{2}a_1 \cdot \xi_1\xi_3 + \eta_1\eta_3 + \zeta_1\zeta_3$$
$$- (h_1 - \tfrac{1}{2}a_2) \cdot - (\xi_2\xi_3 + \eta_2\eta_3 + \zeta_2\zeta_3)$$
$$+ (g_1 - \tfrac{1}{2}a_3) \cdot \xi_3{}^2 + \eta_3{}^2 + \zeta_3{}^2,$$

which is

$$= G \cdot \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

or, as this might be written,

$$= (G, \ F, \ C\,\mathord{\int}a_1, \ h_1, \ g_1) - \tfrac{1}{2}(G, \ F, \ C\,\mathord{\int}a_1, \ a_2, \ a_3).$$

Retaining the former form, the result is

$$L \cdot \square = G \cdot \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

and it is now very easy to write down the complete system of the 18 equations; viz. if the determinant above written down be called $11 \cdot 1 \cdot 2$, and so in other cases, then we have

		A	H	G
$2 \cdot 3$		A	H	G
$3 \cdot 1$		H	B	F
$1 \cdot 2$		G	F	C
$L \cdot 11$	„ $=$	$\tfrac{1}{2}a_1$	$h_1 - \tfrac{1}{2}a_2$	$g_1 - \tfrac{1}{2}a_3$
„ 22	„ $=$	$h_2 - \tfrac{1}{2}b_1$	$\tfrac{1}{2}b_2$	$f_2 - \tfrac{1}{2}b_3$
„ 33	„ $=$	$g_3 - \tfrac{1}{2}c_1$	$f_3 - \tfrac{1}{2}c_2$	$\tfrac{1}{2}c_3$
„ 23	„ $= \tfrac{1}{2}(-f_1 + g_2 + h_3)$		$\tfrac{1}{2}b_3$	$\tfrac{1}{2}c_2$
„ 31	„ $=$	$\tfrac{1}{2}a_3$	$\tfrac{1}{2}(f_1 - g_2 + h_3)$	$\tfrac{1}{2}c_1$
„ 12	„ $=$	$\tfrac{1}{2}a_2$	$\tfrac{1}{2}b_1$	$\tfrac{1}{2}(f_1 + g_2 - h_3)$:

read for instance

$$L \cdot 11 \cdot 1 \cdot 2 = G \cdot \tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3),$$

the equation obtained above.

There are eighteen functions as shown by the following diagram:

	$2 \cdot 3$ $3 \cdot 1$ $1 \cdot 2$
$(22)(33) - (23)^2$	$2 \cdot 3$
$(33)(11) - (31)^2$	$3 \cdot 1$
$(11)(22) - (12)^2$	$1 \cdot 2$
$(31)(12) - (11)(23)$	$2 \cdot 3$
$(12)(23) - (22)(31)$	$3 \cdot 1$
$(23)(31) - (33)(12)$	$1 \cdot 2$,

viz. in any line of the diagram the bracketed duads may belong to any one at pleasure, but all to the same, of the three pairs 2.3, 3.1, 1.2; thus the first line might be

$$(22.3.1)(33.3.1) - (23.3.1)^2,$$

or, instead of the 3.1, we might have 2.3 or 1.2. But of the 18 functions I distinguish 6, viz. those in which for the six lines respectively the pairs are 2.3, 3.1, 1.2, 2.3, 3.1, 1.2, as shown in the diagram. Each of these six functions can be obtained under two different forms, and by equating these we have the equations ($\mathfrak{A}$), ($\mathfrak{B}$), ($\mathfrak{C}$), ($\mathfrak{F}$), ($\mathfrak{G}$), ($\mathfrak{H}$), before referred to; thus ($\mathfrak{C}$) is obtained by equating two different forms of the function $(11.1.2)(22.1.2) - (12.1.2)^2$; and ($\mathfrak{H}$) is obtained by equating two different forms of the function $(23.1.2)(31.1.2) - (33.1.2)(12.1.2)$.

The determinants $11.1.2$, &c., may be denoted by accented letters a, b, c, f, g, h, as follows:

$$
\begin{vmatrix}
x_{11}, & x_{22}, & x_{33}, & x_{23}, & x_{31}, & x_{12} \\
y_{11}, & y_{22}, & y_{33}, & y_{23}, & y_{31}, & y_{12} \\
z_{11}, & z_{22}, & z_{33}, & z_{23}, & z_{31}, & z_{12}
\end{vmatrix}
\begin{matrix}
x_2, & x_3 \\
y_2, & y_3 \\
z_2, & z_3
\end{matrix}
= \text{a}', \ \text{b}', \ \text{c}', \ \text{f}', \ \text{g}', \ \text{h}',
$$

$$
\qquad\qquad\qquad\qquad
\begin{matrix}
x_3, & x_1 \\
y_3, & y_1 \\
z_3, & z_1
\end{matrix}
= \text{a}'', \ \text{b}'', \ \text{c}'', \ \text{f}'', \ \text{g}'', \ \text{h}'',
$$

$$
\qquad\qquad\qquad\qquad
\begin{matrix}
x_1, & x_2 \\
y_1, & y_2 \\
z_1, & z_2
\end{matrix}
= \text{a}''', \ \text{b}''', \ \text{c}''', \ \text{f}''', \ \text{g}''', \ \text{h}''';
$$

viz.

$$
\begin{vmatrix}
x_{11}, & x_2, & x_3 \\
y_{11}, & y_2, & y_3 \\
z_{11}, & z_2, & z_3
\end{vmatrix}
= \text{a}', \ \&\text{c.}
$$

In this notation, ($\mathfrak{C}$) is obtained by equating two values of $\text{a}'''\text{b}''' - (\text{h}''')^2$, and ($\mathfrak{H}$) by equating two values of $\text{f}'''\text{g}''' - \text{c}'''\text{h}'''$.

The forms which I call the second are those given by the immediate substitution of the foregoing values of $(11.1.2)$, &c.; to obtain the first forms, I proceed to calculate those of ($\mathfrak{C}$) and ($\mathfrak{H}$).

Forming by the ordinary rule the product of the determinants $(11.1.2)$ and $(22.1.2)$, which are

$$
\begin{vmatrix}
x_{11}, & y_{11}, & z_{11} \\
x_1, & y_1, & z_1 \\
x_2, & y_2, & z_2
\end{vmatrix}
\quad \text{and} \quad
\begin{vmatrix}
x_{22}, & y_{22}, & z_{22} \\
x_1, & y_1, & z_1 \\
x_2, & y_2, & z_2
\end{vmatrix},
$$

this is

$$\begin{vmatrix} 11.22, & 1.11, & 2.11 \\ 1.22, & 1.1, & 1.2 \\ 2.22, & 1.2, & 2.2 \end{vmatrix},$$

where 11.22 denotes $x_{11}x_{22} + y_{11}y_{22} + z_{11}z_{22}$, and the like for the other symbols. In like manner, the square of the determinant $(12.1.2)$, that is, of

$$\begin{vmatrix} x_{12}, & y_{12}, & z_{12} \\ x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \end{vmatrix},$$

is

$$\begin{vmatrix} 12.12, & 1.12, & 2.12 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 2.1, & 2.2 \end{vmatrix},$$

or, observing that in the two resulting determinants the terms 11.22 and 12.12 are multiplied by the same factor, the expression for the difference gives

$$(11.1.2)(22.1.2) - (12.1.2)^2$$

$$= (11.22 - 12.12)\begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix} + \begin{vmatrix} 0, & 1.11, & 2.11 \\ 1.22, & 1.1, & 1.2 \\ 2.22, & 1.2, & 2.2 \end{vmatrix} - \begin{vmatrix} 0, & 1.12, & 2.12 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 1.2, & 2.2 \end{vmatrix},$$

containing $11.22 - 12.12$, which, by what precedes, is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12});$$

the other terms are also known, viz. the whole value is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2) + \begin{vmatrix} 0, & \tfrac{1}{2}a, & h_1 - \tfrac{1}{2}a_2 \\ h_2 - \tfrac{1}{2}b_1, & a, & h \\ \tfrac{1}{2}b_2, & h, & b \end{vmatrix} - \begin{vmatrix} 0, & \tfrac{1}{2}a_2, & \tfrac{1}{2}b_1 \\ \tfrac{1}{2}a_2, & a, & h \\ \tfrac{1}{2}b_1, & h, & b \end{vmatrix},$$

which is

$$= -\tfrac{1}{2}(a_{22} + b_{11} - 2h_{12})(ab - h^2)$$
$$+ a(\tfrac{1}{4}b_1{}^2 - \tfrac{1}{2}b_2h_1 + \tfrac{1}{4}a_2b_2)$$
$$+ b(\tfrac{1}{4}a_2{}^2 - \tfrac{1}{2}a_1h_2 + \tfrac{1}{4}a_1b_1)$$
$$+ h(\tfrac{1}{4}a_1b_2 - \tfrac{1}{4}a_2b_1 + h_1h_2 - \tfrac{1}{2}a_2h_2 - \tfrac{1}{2}b_1h_1),$$

$= (ab - h^2)k$, where k is the measure of curvature.

In the same way, we have

$$(23.1.2)(31.1.2) - (33.1.2)(12.1.2)$$

$$= \begin{vmatrix} 23.31, & 1.31, & 2.31 \\ 1.23, & 1.1, & 1.2 \\ 2.23, & 1.2, & 2.2 \end{vmatrix} - \begin{vmatrix} 12.33, & 1.33, & 2.33 \\ 1.12, & 1.1, & 1.2 \\ 2.12, & 1.2, & 2.2 \end{vmatrix},$$

which is

$$= (23.31 - 12.33) \begin{vmatrix} 1.1, & 1.2 \\ 1.2, & 2.2 \end{vmatrix} + \begin{vmatrix} 0 & , & 1.13, & 2.13 \\ 1.23, & 1.1 & , & 1.2 \\ 2.23, & 1.2 & , & 2.2 \end{vmatrix} - \begin{vmatrix} 0 & , & 1.33, & 2.33 \\ 1.12, & 1.1 & , & 1.2 \\ 2.12, & 1.2 & , & 2.2 \end{vmatrix},$$

containing $23.31 - 12.33$, which by what precedes is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33});$$

the other terms are also known, and the value is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$

$$+ \begin{vmatrix} 0 & , & \tfrac{1}{2}a_3, & \tfrac{1}{2}(f_1 - g_2 + h_3) \\ \tfrac{1}{2}(-f_1 + g_2 + h_3), & a & , & h \\ \tfrac{1}{2}b_3 & , & h & , & b \end{vmatrix} - \begin{vmatrix} 0 & , & g_3 - \tfrac{1}{2}c_1, & f_3 - \tfrac{1}{2}c_2 \\ \tfrac{1}{2}a_2, & a & , & h \\ \tfrac{1}{2}b_1, & h & , & b \end{vmatrix},$$

or finally this is

$$= -\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$

$$+ a\left(\tfrac{1}{2}b_1 f_3 - \tfrac{1}{4}b_3 f_1 - \tfrac{1}{4}b_1 c_2 + \tfrac{1}{4}b_3 g_2 - \tfrac{1}{4}b_3 h_3\right)$$

$$+ b\left(\tfrac{1}{2}a_2 g_3 - \tfrac{1}{4}a_3 g_2 - \tfrac{1}{4}a_2 c_1 + \tfrac{1}{4}a_3 f_1 - \tfrac{1}{4}a_3 h_3\right)$$

$$+ h\left\{\tfrac{1}{4}a_3 b_3 + \tfrac{1}{4}b_1 c_1 + \tfrac{1}{4}a_2 c_2 - \tfrac{1}{2}b_1 g_3 - \tfrac{1}{2}a_2 f_3 + \tfrac{1}{4}h_3^2 - \tfrac{1}{4}(f_1 - g_2)^2\right\}.$$

The remaining four of the six functions can of course be obtained from the two just found by a cyclical interchange of the letters and suffix-numbers 1, 2, 3, and it is not worth while to write down the values.

The two values of $(11.1.2)(22.1.2) - (12.1.2)^2$ are

$$-(a_{22} + b_{11} - 2h_{12})(ab - h^2) + a\left(\tfrac{1}{4}b_1^2 - \tfrac{1}{2}b_2 h_1 + \tfrac{1}{4}a_2 b_2\right)$$

$$+ b\left(\tfrac{1}{4}a_2^2 - \tfrac{1}{2}a_1 h_2 + \tfrac{1}{4}a_1 b_1\right)$$

$$+ h\left\{\tfrac{1}{4}(a_1 b_2 - a_2 b_1) + h_1 h_2 - \tfrac{1}{2}b_1 h_1 - \tfrac{1}{2}a_2 h_2\right\},$$

and

$$\left\{G.\tfrac{1}{2}a_1 + F(h_1 - \tfrac{1}{2}a_2) + C(g_1 - \tfrac{1}{2}a_3)\right\}.\left\{G(h_2 - \tfrac{1}{2}b_1) + F.\tfrac{1}{2}b_2 + C(f_2 - \tfrac{1}{2}b_3)\right\}$$

$$- \left\{G.\tfrac{1}{2}a_2 + F.\tfrac{1}{2}b_1 + C.\tfrac{1}{2}(f_1 + g_2 - h_3)\right\}^2,$$

where, in the first value, for a, b, h, $ab - h^2$ we must write $L^{-2}(BC - F^2)$, $L^{-2}(CA - G^2)$, $L^{-2}(FG - CH)$, and L^2C; making this change, multiplying by 4 and equating, we obtain

$$- 2L^2C(a_{22} + b_{11} - 2h_{12})$$

$$+ (BC - F^2)(b_1^2 - 2b_2 h_1 + a_2 b_2)$$

$$+ (CA - G^2)(a_2^2 - 2a_1 h_2 + a_1 b_1)$$

$$+ (FG - CH)(a_1 b_2 - a_2 b_1 + 4h_1 h_2 - 2b_1 h_1 - 2a_2 h_2)$$

$$- \left\{Ga_1 + F(2h_1 - a_2) + C(2g_1 - a_3)\right\}\left\{G(2h_2 - b_1) + Fb_2 + C(2f_2 - b_3)\right\}$$

$$+ \left\{Ga_2 + Fb_1 + C(f_1 + g_2 - h_3)\right\}^2 \qquad = 0.$$

C. XII.   2

Developing the fourth and fifth lines, it appears that in this expression the coefficients of F^2, G^2 and FG each of them vanish; the whole equation is thus divisible by C, and omitting this factor throughout, the equation becomes

$$
\begin{aligned}
0 = & - 2L^2 (a_{22} + b_{11} - 2h_{12}) \\
& + A\, (a_2^2 - 2a_1 h_2 + a_1 b_1) \\
& + B\, (b_1^2 - 2b_2 h_1 + a_2 b_2) \\
& + C\, \{- a_3 b_3 + 2a_3 f_2 + 2b_3 g_1 - 4f_2 g_1 + (f_1 + g_2 - h_3)^2\} \\
& + F\, \{(a_3 b_2 - a_2 b_3) + 2(b_1 g_2 - b_2 g_1) + 2(b_3 h_1 - b_1 h_3) + 2(a_2 f_2 + b_1 f_1 - 2h_1 f_2)\} \\
& + G\, \{a_1 b_3 - a_3 b_1 + 2(a_2 f_1 - a_1 f_2) + 2(a_3 h_2 - a_2 h_3) + 2(a_2 g_2 + b_1 g_1 - 2g_1 h_2)\} \\
& + H\, \{\qquad\qquad\qquad\qquad\qquad 2(a_2 h_2 + b_1 h_1 - 2h_1 h_2)\}\,;
\end{aligned}
$$

this is substantially the required equation (ℭ), but the form of it may be greatly simplified.

Forming the identity,

$$
\begin{aligned}
0 = 2L[(a_2 - h_1)L_2 + (b_1 - h_2)L_1] = & - A\,(a_1 b_1 - a_1 h_2 - a_2 h_1 + a_2^2) \\
& - B\,(b_1^2 - b_1 h_2 - b_2 h_1 + a_2 b_2) \\
& - C\,(b_1 c_1 + a_2 c_2 - c_1 h_2 - c_2 h_1) \\
& - 2F\,(b_1 f_1 + a_2 f_2 - f_1 h_2 - f_2 h_1) \\
& - 2G\,(b_1 g_1 + a_2 g_2 - g_1 h_2 - g_2 h_1) \\
& - 2H\,(a_2 h_2 + b_1 h_1 - 2h_1 h_2),
\end{aligned}
$$

we add hereto the last preceding equation; the coefficients of A, B, F, G, H thus assume new and simple forms, but the coefficient of C requires a further transformation.

Assume

$$
\begin{aligned}
\Omega = & (b_1 c_1 - f_1^2) + (c_2 a_2 - g_2^2) + (a_3 b_3 - h_3^2) \\
& + (g_2 h_3 + g_3 h_2 - a_2 f_3 - a_3 f_2) + (h_3 f_1 + h_1 f_3 - b_1 g_3 - b_3 g_1) + (f_1 g_2 + f_2 g_1 - c_1 h_2 - c_2 h_1)\,;
\end{aligned}
$$

then, if we add to the equation C multiplied by this value and subtract $C\Omega$, the coefficient of C takes its proper form, and the equation is

$$
\begin{aligned}
& - 2L^2 (a_{22} + b_{11} - 2h_{12}) \\
& + 2L[(b_1 - h_2)L_1 + (a_2 - h_1)L_2] - C\Omega \\
& + A\, \{-(ah_2 - a_2 h)\} \\
& + B\, \{\ \ (b_1 h_2 - b_2 h_1)\} \\
& + C\, \{-(a_2 f_3 - a_3 f_2) + (b_3 g_1 - b_1 g_3) - (g_2 h_3 - g_3 h_2) - (h_3 f_1 - h_1 f_3) + 3(f_1 g_2 - f_2 g_1)\} \\
& + F\, \{-(a_2 b_3 - a_3 b_2) + 2(b_1 g_2 - b_2 g_1) + 2(b_3 h_1 - b_1 h_3) - 2(h_1 f_2 - h_2 f_1)\} \\
& + G\, \{-(a_3 b_1 - a_1 b_3) - 2(a_1 f_2 - a_2 f_1) - 2(a_2 h_3 - a_3 h_2) - 2(g_1 h_2 - g_2 h_1)\} \\
& + H\, \{-(a_1 b_2 - a_2 b_1)\} \\
& = 0\,;
\end{aligned}
$$

where, as a verification, observe that the letters (a, b), (f, g) may be interchanged if at the same time we interchange the suffixes 1 and 2.

The two values of $(23 \cdot 1 \cdot 2)(31 \cdot 1 \cdot 2) - (33 \cdot 1 \cdot 2)(12 \cdot 1 \cdot 2)$ are

$$-\tfrac{1}{2}(g_{23} + f_{31} - c_{12} - h_{33})(ab - h^2)$$
$$+ a\,(\tfrac{1}{2}b_1f_3 - \tfrac{1}{4}b_3f_1 - \tfrac{1}{4}b_1c_2 + \tfrac{1}{4}b_3g_2 - \tfrac{1}{4}b_3h_3)$$
$$+ b\,(\tfrac{1}{2}a_2g_3 - \tfrac{1}{4}a_3g_2 - \tfrac{1}{4}a_2c_1 + \tfrac{1}{4}a_3f_1 - \tfrac{1}{4}a_3h_3)$$
$$+ h\,\{\tfrac{1}{4}a_3b_3 + \tfrac{1}{4}b_1c_1 + \tfrac{1}{4}a_2c_2 - \tfrac{1}{2}b_1g_3 - \tfrac{1}{2}a_2f_3 + \tfrac{1}{4}h_3{}^2 - \tfrac{1}{4}(f_1 - g_2)^2\},$$

and

$$\{G \cdot \tfrac{1}{2}(-f_1 + g_2 + h_3) + F \cdot \tfrac{1}{2}b_3 + C \cdot \tfrac{1}{2}c_2\}\,\{G \cdot \tfrac{1}{2}a_3 + F \cdot \tfrac{1}{2}(f_1 - g_2 + h_3) + C \cdot \tfrac{1}{2}c_1\}$$
$$- \{G\,(g_3 - \tfrac{1}{2}c_1) + F\,(f_3 - \tfrac{1}{2}c_2) + C \cdot \tfrac{1}{2}c_3\}\,\{G \cdot \tfrac{1}{2}a_2 + F \cdot \tfrac{1}{2}b_1 + C \cdot \tfrac{1}{2}(f_1 + g_2 - h_3)\},$$

and here for a, b, h, $ab - h^2$, substituting their values, multiplying by 4, and equating, we have

$$-2L^2C\,(g_{23} + f_{31} - c_{12} - h_{33})$$
$$+ (BC - F^2)\,(2b_1f_3 - b_3f_1 - b_1c_2 + b_3g_2 - b_3h_3)$$
$$+ (CA - G^2)\,(2a_2g_3 - a_3g_2 - a_2c_1 + a_3f_1 - a_3h_3)$$
$$+ (FG - CH)\,(a_3b_3 + b_1c_1 + a_2c_2 - 2b_1g_3 - 2a_2f_3 + h_3{}^2 - f_1{}^2 + 2f_1g_2 - g_2{}^2)$$
$$- \{G\,(-f_1 + g_2 + h_3) + F b_3 + C c_2\}\,\{G a_3 + F\,(f_1 - g_2 + h_3) + C c_1\}$$
$$+ \{G\,(2g_3 - c_1) + F\,(2f_3 - c_2) + C c_3\}\,\{G a_2 + F b_1 + C\,(f_1 + g_2 - h_3)\} = 0.$$

Here again the terms in F^2, FG, G^2 all vanish, hence the whole equation divides by C; throwing this factor out, we have

$$-2L^2\,(g_{23} + f_{31} - c_{12} - h_{33})$$
$$+ A\,(\quad 2a_2g_3 - a_3g_2 \quad - \quad a_2c_1 + \quad a_3f_1 \quad - \quad a_3h_3)$$
$$+ B\,(\quad 2b_1f_3 \quad - b_3f_1 \quad - \quad b_1c_2 + \quad b_3g_2 - \quad b_3h_3)$$
$$+ C\,(- \quad c_1c_2 + c_3f_1 \quad + \quad c_3g_2 - \quad c_3h_3)$$
$$+ F\,\{-(b_3c_1 - b_1c_3) - 2c_2f_1 + 2f_1f_3 \quad + 2g_2f_3 - 2f_3h_3\}$$
$$+ G\,(\quad a_2c_3 - a_3c_2 \quad - 2c_1g_2 + 2f_1g_3 \quad + 2g_2g_3 - 2g_3h_3)$$
$$+ H\,(- a_3b_3 - b_1c_1 \quad - \quad a_2c_2 + 2b_1g_3 + 2a_2f_3 - h_3{}^2 + f_1{}^2 - 2f_1g_2 + g_2{}^2) = 0,$$

which is substantially the required equation $(\mathfrak{H})$; but the form has to be altered. First, multiply by 2, then forming the expression

$$2L\,[(f_3 - c_2)\,L_1 + (g_3 - c_1)\,L_2 + (f_1 + g_2 - 2h_3)\,L_3]$$
$$= \quad (f_3 - \quad c_2)\,(A a_1 + B b_1 + C c_1 + 2F f_1 + 2G g_1 + 2H h_1)$$
$$+ (g_3 - \quad c_1)\,(A a_2 + B b_2 + C c_2 + 2F f_2 + 2G g_2 + 2H h_2)$$
$$+ (f_1 + \quad g_2 - 2h_3)\,(A a_3 + B b_3 + C c_3 + 2F f_3 + 2G g_3 + 2H h_3),$$

this is

$$
\begin{aligned}
= \ & A \left(- a_1 c_2 - a_2 c_1 + a_1 f_3 + a_3 f_1 + a_2 g_3 + a_3 g_2 - 2 a_3 h_3 \right) \\
+ \ & B \left(- b_1 c_2 - b_2 c_1 + b_1 f_3 + b_3 f_1 + b_2 g_3 + b_3 g_2 - 2 b_3 h_3 \right) \\
+ \ & C \left(- 2 c_1 c_2 \qquad\quad + c_1 f_3 + c_3 f_1 + c_2 g_3 + c_3 g_2 - 2 c_3 h_3 \right) \\
+ \ & 2F \left(- c_1 f_2 - c_2 f_1 + 2 f_1 f_3 + \qquad\quad f_2 g_3 + f_3 g_2 - 2 f_3 h_3 \right) \\
+ \ & 2G \left(- c_1 g_2 - c_2 g_1 + f_1 g_3 + f_3 g_1 + 2 g_2 g_3 - 2 g_3 h_3 \right) \\
+ \ & 2H \left(- c_1 h_2 - c_2 h_1 + g_2 h_3 + g_3 h_2 + h_3 f_1 + h_1 f_3 - 2 h_3{}^2 \right);
\end{aligned}
$$

then adding this expression

$$
(f_3 - c_2) L_1 + (g_3 - c_1) L_2 + (f_1 + g_2 - 2h_3) L_3
$$

and subtracting its foregoing value, we have an equation with new coefficients of A, B, C, F, G, H, all of which, except that of H, are in the proper form, and for the coefficient of H, recurring to the foregoing value of Ω, we must add to the equation $2H$ multiplied by the value of Ω and subtract $2H\Omega$. The final result is

$$
\begin{aligned}
& -4 L^2 \left(g_{23} + f_{31} - c_{12} - h_{33} \right) \\
& + 2L \left[(f_3 - c_2) L_1 + (g_3 - c_1) L_2 + (f_1 + g_2 - 2h_3) L_3 \right] - 2H\Omega \\
& + \ A \ \{ \quad 3 \left(a_2 g_3 - a_3 g_2 \right) - \left(c_1 a_2 - c_2 a_1 \right) + \left(a_3 f_1 - a_1 f_3 \right) \} \\
& + \ B \ \{ - 3 \left(b_3 f_1 - b_1 f_3 \right) - \left(b_1 c_2 - b_2 c_1 \right) - \left(b_2 g_3 - b_3 g_2 \right) \} \\
& + \ C \ \{ \quad \left(c_3 f_1 - c_1 f_3 \right) - \left(c_2 g_3 - c_3 g_2 \right) \qquad\qquad \} \\
& + \ 2F \ \{ - \left(b_3 c_1 - b_1 c_3 \right) + \left(c_1 f_2 - c_2 f_1 \right) - \left(f_2 g_3 - f_3 g_2 \right) \} \\
& + \ 2G \ \{ - \left(c_2 a_3 - c_3 a_2 \right) - \left(c_1 g_2 - c_2 g_1 \right) - \left(f_3 g_1 - f_1 g_3 \right) \} \\
& + \ 2H \ \{ - \left(b_3 g_1 - b_1 g_3 \right) + \left(a_2 f_3 - a_3 f_2 \right) \qquad\qquad \} = 0,
\end{aligned}
$$

which, it will be observed, remains unaltered by the interchange of a and b, f and g A and B, F and G, the suffix numbers 1 and 2 being at the same time interchanged.

We have thus the required six equations, in which

$$
L^2 = abc - af^2 - bg^2 - ch^2 + 2fgh,
$$

$$
\Omega = b_1 c_1 - f_1{}^2 + c_2 a_- - g_2{}^2 + a_3 b_3 - h_3{}^2
$$

$$
+ (g_2 h_3 + g_3 h_2 - a_2 f_3 - a_3 f_2) + (h_3 f_1 + h_1 f_3 - b_3 g_1 - b_1 g_3) + (f_1 g_2 + f_2 g_1 - a_1 f_2 - a_2 f_1),
$$

and where I write also, for shortness, ab12 for $a_1 b_2 - a_2 b_1$, &c.; viz. the equations are

$$
\begin{aligned}
0 = & - 2L^2 \left(b_{33} + c_{22} - 2 f_{23} \right) && (\mathfrak{A}). \\
& + 2L \ [\quad . \quad (c_2 - f_3) L_2 + (b_3 - f_2) L_3] && \qquad - A\Omega \\
& + \ A \ \{ \quad . \quad - \ bg31 + \quad ch12 + 3 . gh23 - hf31 - \quad fg12 \} \\
& + \ B \ \{ - bf23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \\
& + \ C \ \{ + cf23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \\
& + \ F \ \{ - bc23 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \} \\
& + \ G \ \{ - bc31 + 2 . cf12 \quad . \quad + 2 . ch23 \quad . \quad - 2 . fg23 \} \\
& + \ H \ \{ - bc12 - 2 . bf31 - 2 . bg23 \quad . \quad . \quad - 2 . hf23 \}
\end{aligned}
$$

$$0 = - 2L^2(c_{11} + a_{33} - 2g_{31}) \qquad\qquad (\mathfrak{B}).$$
$$+ \ 2L\ [(c_1 - g_3)\, L_1 \qquad . \qquad\qquad + (a_3 - g_1)\, L_3] \qquad\qquad - B\Omega$$
$$+ \ A\ \{ \qquad . \qquad\quad ag31 \qquad\qquad\qquad\qquad\qquad \}$$
$$+ \ B\ \{ \quad af23 \qquad . \qquad - \quad ch12 - \quad gh23 + 3\,.\,hf31 - \quad fg31\}$$
$$+ \ C\ \{ \qquad - \quad cg31 \qquad\qquad\qquad\qquad\qquad\qquad \}$$
$$+ \ F\ \{ - ca23 \qquad . \qquad\qquad\quad - 2\,.\,cg12 - 2\,.\,ch31 - 2\,.\,fg31\}$$
$$+ \ G\ \{ - ca31 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \}$$
$$+ \ H\ \{ - ca12 + 2\,.\,af31 + 2\,.\,ag23 \qquad\qquad\qquad 2\,.\,gh31\}$$

$$0 = - 2L^2(a_{22} + b_{11} - 2h_{12}) \qquad\qquad (\mathfrak{C}).$$
$$+ \ 2L\ [(b_1 - h_2)\, L_1 + (a_2 - h_1)\, L_2 \qquad . \qquad] \qquad\qquad - C\Omega$$
$$+ \ A\ \{ \qquad . \qquad\qquad - \qquad ah12 \qquad\qquad\qquad\quad \}$$
$$+ \ B\ \{ \qquad . \qquad\qquad + \qquad bh12 \qquad\qquad\qquad\quad \}$$
$$+ \ C\ \{ - af23 + bg31 \qquad . \qquad - \quad gh23 - \qquad hf31 + 3\,.\,fg12\}$$
$$+ \ F\ \{ - ab23 \qquad . \qquad\qquad\quad + 2\,.\,bg12 + 2\,.\,bh31 - 2\,.\,hf12\}$$
$$+ \ G\ \{ - ab31 \qquad . \qquad - 2\,.\,af12 \qquad . \qquad\quad + 2\,.\,ah23 - 2\,.\,gh12\}$$
$$+ \ H\ \{ - ab12 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \}$$

$$0 = - 4L^2(g_{12} + h_{31} - a_{23} - f_{11}) \qquad\qquad (\mathfrak{F}).$$
$$+ \ 2L\ [(g_2 + h_3 - 2f_1)\, L_1 + (g_1 - a_3)\, L_2 + (h_1 - a_2)\, L_3] \quad - 2F\Omega$$
$$+ \ A\ \{ \qquad . \qquad\qquad ag12 - \qquad ah31 \qquad\quad \}$$
$$+ \ B\ \{ - ab23 + \qquad bg12 + 3\,.\,bh31 \qquad\qquad \}$$
$$+ \ C\ \{ - ca23 - 3\,.\,cg12 - \qquad ch31 \qquad\qquad \}$$
$$+ \ 2F\ \{ \quad . \qquad . \qquad + \qquad bg31 - \qquad ch12 \qquad\qquad \}$$
$$+ \ 2G\ \{ - ca12 + \qquad ag23 \qquad . \qquad - gh31\}$$
$$+ \ 2H\ \{ - ab31 \qquad\qquad . \qquad - \quad ah23 - gh12\}$$

$$0 = - 4L^2(h_{23} + f_{12} - b_{31} - g_{22}) \qquad\qquad (\mathfrak{G}).$$
$$+ \ 2L\ [(f_2 - b_3)\, L_1 + (h_3 + f_1 - 2g_2)\, L_2 + (h_2 - b_1)\, L_3] \quad - 2G\Omega$$
$$+ \ A\ \{ - ab31 - 3\,.\,ah23 - \qquad af12 \qquad\qquad \}$$
$$+ \ B\ \{ \qquad . \qquad\quad bh23 - \qquad bf12 \qquad\qquad \}$$
$$+ \ C\ \{ - bc31 + \qquad ch23 + 3\,.\,cf12 \qquad\qquad \}$$
$$+ \ 2F\ \{ - bc12 \qquad . \qquad - \qquad bf31 - hf23\}$$
$$+ \ 2G\ \{ \qquad . \qquad + \quad ch12 - \qquad af23 \qquad . \qquad \}$$
$$+ \ 2H\ \{ - ab23 + \qquad bh31 \qquad . \qquad - hf12\}$$

$$0 = - 4L^2 (f_{31} + g_{23} - c_{12} - h_{33}) \qquad\qquad (\mathfrak{H}).$$

$$+ 2L \left[(f_3 - c_2) L_1 + (g_3 - c_1) L_2 + (f_1 + g_2 - 2h_3) L_3\right] \; - 2H\Omega$$

$$+ \; A \left\{- \text{ca}12 + \quad \text{af}31 + 3 . \text{ag}23 \qquad\qquad\right\}$$

$$+ \; B \left\{- \text{bc}12 - 3 . \text{bf}31 - \quad \text{bg}23 \qquad\qquad\right\}$$

$$+ \; C \left\{ \quad . \qquad \text{cf}31 - \quad \text{cg}23 \cdot \qquad\qquad\right\}$$

$$+ \; F \left\{- \text{bc}31 + \quad \text{cf}12 \qquad . \qquad - \text{fg}23\right\}$$

$$+ \; G \left\{- \text{ca}23 \qquad . \qquad - \text{cg}12 - \text{fg}31\right\}$$

$$+ \; H \left\{ \quad . \quad + \quad \text{af}23 - \quad \text{bg}31 \quad . \qquad\right\}$$

It would be possible in these equations to introduce the symbols $AB12$, &c., in place of ab12, &c., and then writing $A = B = C = 1$, all these symbols other than those where the letters are GH, HF or FG would vanish, and we should obtain Mr Warren's six equations for normal coordinates. But in the general case it would seem that there is not any advantage in the introduction of the new symbols $AB12$, &c., and I retain by preference the equations in the form in which I have given them.

To the foregoing may be joined a symmetrical equation obtained (as by Mr Warren) by multiplying the several equations by a, b, c, f, g, h respectively, and adding; the result is in the first instance obtained in the form

$$- 2L^2\Theta + 2L\Psi - 3L^2\Omega + \square = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathfrak{M}),$$

where

$$\Theta = \quad \text{a} \, (\text{b}_{33} + \text{c}_{22} - 2\text{f}_{23})$$

$$+ \; \text{b} \, (\text{c}_{11} + \text{a}_{33} - 2\text{g}_{31})$$

$$+ \; \text{c} \, (\text{a}_{22} + \text{b}_{11} - 2\text{h}_{12})$$

$$+ \; 2\text{f} \, (\text{g}_{12} + \text{h}_{31} - \text{a}_{23} - \text{f}_{11})$$

$$+ \; 2\text{g} \, (\text{h}_{23} + \text{f}_{12} - \text{b}_{31} - \text{g}_{22})$$

$$+ \; 2\text{h} \, (\text{f}_{31} + \text{g}_{23} - \text{c}_{12} - \text{h}_{33}).$$

For Ψ, collecting the terms which contain L_1, L_2, L_3 respectively, and attending to the values of (A, B, C, F, G, H),

$$= (\text{bc} - \text{f}^2, \; \text{ca} - \text{g}^2, \; \text{ab} - \text{h}^2, \; \text{gh} - \text{af}, \; \text{hf} - \text{bg}, \; \text{fg} - \text{ch}),$$

this is easily reduced to the form

$$\Psi = (A_1 + H_2 + G_3) L_1 + (H_1 + B_2 + F_3) L_2 + (G_1 + F_2 + C_3) L_3.$$

The term in Ω is

$$= - (A\text{a} + B\text{b} + C\text{c} + 2F\text{f} + 2G\text{g} + 2H\text{h}) \, \Omega,$$

$$= - 3L^2\Omega, \text{ as above.}$$

For the calculation of $\square$, collecting the terms, we have $\square =$

23	A	B	C	F	G	H	31	A	B	C	F	G	H	12	A	B	C	F	G	H
bc				− a			bc			− g	− 2h	− a		bc		− h		− 2g		− a
ca		− f	− b	− 2h			ca					− b		ca	− h				− 2f	− b
ab		− f	− c		− 2g		ab	− g			− c	− 2f		ab						− c
af		b	− c		− 2g	2h	af	h				2b		af	− g				− 2c	
bf		− a					bf			− 3h	− 2g	− 2a		bf		− g				
cf			a				cf			h				cf			3g	2h	2a	
ag	3h				2f	2b	ag	b						ag	f					
bg		− h				− 2a	bg	− a		+ c	2f		− 2h	bg		f			2c	
cg			− h				cg			− b				cg			− 3f	− 2b	− 2h	
ah	− 3g			− 2c	− 2f		ah	− f						ah	− c					
bh		g					bh			3f	2c	2g		bh		c				
ch			g		2a		ch			− f	− 2b			ch	a	− b		− 2f	2g	
gh	− 3a	− b	− c				gh					− 2f	− 2b	gh					− 2c	− 2f
hf					− 2g	− 2a	hf	− a	3b	− c				hf					− 2c	− 2g
fg					− 2h	− 2a	fg				− 2b	− 2h		fg	− a	− b	+ 3c			

viz. this is

$$\square = bc23 . - aF + bc31\,(- gC - 2hF - aG) + bc12\,(- hB - 2gF - aH) + \&c.$$

Some however of the coefficients require reduction; for instance, that of bc31 is $= -(a, h, g\,\rangle\,G, F, C) - hF, = -hF$; and so $- hB - 2gF - aH$ is $= -(a, h, g\,\rangle\,H, B, F) - gF, = -gF$. After these reductions, the value is found to be $\square =$

bc23 . − aF	+ bc31 . − hF	+ bc12 . − gF
+ ca23 . − hG	+ ca31 . − bG	+ ca12 . − fG
+ ab23 . − gH	+ ab31 . − fH	+ ab12 . − cH
+ af23 (− gG + hH)	+ af31 (bH − fG)	+ af12 (fH − cG)
+ bf23 . − aB	+ bf31 . − hB	+ bf12 . − gB
+ cf23 . aC	+ cf31 . hC	+ cf12 . gC
+ ag23 . hA	+ ag31 . bA	+ ag12 . fA
+ bg23 (gF − aH)	+ bg31 (− hH + fF)	+ bg12 (cF − gH)
+ cg23 . − hC	+ cg31 . − bC	+ cg12 . − fC
+ ah23 . − gA	+ ah31 . − fA	+ ah12 . − cA
+ bh23 . gB	+ bh31 . fB	+ bh12 . cB
+ ch23 (aG − hF)	+ ch31 (hG − bF)	+ ch12 (− fF + gG)
+ gh23 (3aA − bB − cC)	+ gh31 . 2hA	+ gh12 . 2gA
+ hf23 . 2hB	+ hf31 (− aA + 3bB − cC)	+ hf12 . 2fB
+ fg23 . 2gC	+ fg31 . 2fC	÷ fg12 (− aA − bB + 3cC);

or, arranging the terms in a different order, this may be written

$\square =$

$A \{$ $a\,(3\,.\,\mathrm{gh}23 - \mathrm{hf}31 - \mathrm{fg}12) + b\,.\,\mathrm{ag}31 - c\,.\,\mathrm{ah}12 \qquad\qquad - c\,.\,\mathrm{ah}12$

$\qquad\qquad + f\,(\mathrm{ag}12 - \mathrm{ah}31) \quad + g\,(2\,.\,\mathrm{gh}12 - \mathrm{ah}23) + h\,(2\,.\,\mathrm{gh}31 + \mathrm{ag}23)\}$

$+ B \{- a\,.\,\mathrm{bf}23 \qquad\qquad\qquad + b\,(-\mathrm{gh}23 + 3\,.\,\mathrm{hf}31 - \mathrm{fg}12) + c\,.\,\mathrm{bh}12$

$\qquad\qquad + f\,(2\,.\,\mathrm{hf}12 + \mathrm{bh}31) + g\,(\mathrm{bh}23 - \mathrm{bf}12) \quad + h\,(2\,.\,\mathrm{hf}23 - \mathrm{bf}31)\}$

$+ C \{$ $a\,.\,\mathrm{cf}23 \qquad\qquad\qquad - b\,.\,\mathrm{cg}31 \qquad\qquad\qquad + c\,(-\mathrm{gh}23 - \mathrm{hf}31 + 3\,.\,\mathrm{fg}12)$

$\qquad\qquad + f\,(2\,.\,\mathrm{fg}31 - \mathrm{cg}12) \quad + g\,(2\,.\,\mathrm{fg}23 + \mathrm{cf}12) \quad + h\,(\mathrm{cf}31 - \mathrm{cg}23)\}$

$+ F \{- a\,.\,\mathrm{bc}23 \qquad\qquad\qquad - b\,.\,\mathrm{ch}31 \qquad\qquad\qquad + c\,.\,\mathrm{bg}12$

$\qquad\qquad + f\,(\mathrm{bg}31 - \mathrm{ch}12) \quad + g\,(\mathrm{bg}23 - \mathrm{bc}12) \quad - h\,(\mathrm{ch}23 + \mathrm{bc}31)\}$

$+ G \{+ a\,.\,\mathrm{ch}23 \qquad\qquad\qquad - b\,.\,\mathrm{ca}31 \qquad\qquad\qquad - c\,.\,\mathrm{af}12$

$\qquad\qquad - f\,(\mathrm{af}31 + \mathrm{ca}12) \quad + g\,(\mathrm{ch}12 - \mathrm{af}23) \quad + h\,(\mathrm{ch}31 - \mathrm{ca}23)\}$

$+ H \{- a\,.\,\mathrm{bg}23 \qquad\qquad\qquad + b\,.\,\mathrm{af}31 \qquad\qquad\qquad - c\,.\,\mathrm{ab}12$

$\qquad\qquad + f\,(\mathrm{af}12 - \mathrm{ab}31) \quad - g\,(\mathrm{bg}12 + \mathrm{ab}23) \quad + h\,(\mathrm{af}23 - \mathrm{bg}31)\}.$

Attending to the values of A, B, C, F, G, H, we have

$$A_{11} + B_{22} + C_{33} + 2F_{23} + 2G_{31} + 2H_{12}$$

$$= \quad \mathrm{bc}_{11} + \mathrm{cb}_{11} - 2\mathrm{ff}_{11} \qquad + 2\,(b_1 c_1 - f_1{}^2)$$

$$+ \quad \mathrm{ca}_{22} + \mathrm{ac}_{22} - 2\mathrm{gg}_{22} \qquad + 2\,(c_2 a_2 - g_2{}^2)$$

$$+ \quad \mathrm{ab}_{33} + \mathrm{ba}_{33} - 2\mathrm{hh}_{33} \qquad + 2\,(a_3 b_3 - h_3{}^2)$$

$$+ 2\,\{\mathrm{gh}_{23} + \mathrm{hg}_{23} - \mathrm{af}_{23} - \mathrm{fa}_{23} + \quad g_2 h_3 + g_3 h_2 - a_2 f_3 - a_3 f_2\,\}$$

$$+ 2\,\{\mathrm{hf}_{31} + \mathrm{fh}_{31} - \mathrm{bg}_{31} - \mathrm{gb}_{31} + \quad h_3 f_1 + h_1 f_3 - b_3 g_1 - b_1 g_3\}$$

$$+ 2\,\{\mathrm{fg}_{12} + \mathrm{gf}_{12} - \mathrm{ch}_{12} - \mathrm{hc}_{12} + \quad f_1 g_2 + f_2 g_1 - c_1 h_2 - c_2 h_1\},$$

which is, in fact, $= 2\Theta + 2\Omega$.

Hence the foregoing equation ($\mathfrak{M}$) may also be written

$$- L^2\,(A_{11} + B_{22} + C_{33} + 2F_{23} + 2G_{31} + 2H_{12}) + 2L\Psi - L^2\Omega + \square = 0,$$

where Ψ, Ω, $\square$ have their before-mentioned values.

In the particular case where $f = 0$, $g = 0$, $h = 0$, we have

$$A,\ B,\ C,\ F,\ G,\ H = \mathrm{bc},\ \mathrm{ca},\ \mathrm{ab},\ 0,\ 0,\ 0\ ;\quad L^2 = \mathrm{abc}\ ;$$

$$\Omega = b_1 c_1 + c_2 a_2 + a_3 b_3\ ;$$

the equation ($\mathfrak{A}$) becomes

$$- 2\mathrm{abc}\,(b_{33} + c_{22}) + [c_2\,(a_2 \mathrm{bc} + b_2 \mathrm{ca} + c_2 \mathrm{ab}) + b_3\,(a_3 \mathrm{bc} + b_3 \mathrm{ca} + c_3 \mathrm{ab})] - \mathrm{bc}\,(b_1 c_1 + c_2 a_2 + a_3 b_3) = 0,$$

that is,

$$- 2\mathrm{abc}\,(b_{33} + c_{22}) - \mathrm{bc}\,b_1 c_1 + \mathrm{ca}\,(b_2 c_2 + b_3{}^2) + \mathrm{ab}\,(b_3 c_3 + c_2{}^2) = 0,$$

and the equation ($\mathfrak{F}$) becomes

$$4abc \cdot a_{23} - [a_3 (a_2 bc + b_2 ca + c_2 ab) + a_2 (a_3 bc + b_3 ca + c_3 ab)]$$
$$- ca (a_2 b_3 - a_3 b_2) - ab (c_2 a_3 - c_3 a_2) = 0,$$

that is,

$$4abc \cdot a_{23} - 2bc \cdot a_2 a_3 - 2ca \cdot a_2 b_3 - 2ab \cdot a_3 b_2 = 0.$$

Dividing by $-2abc$ and $4abc$ respectively, and completing the system, the six equations are

$$b_{33} + c_{22} - \frac{b_3^2}{2b} - \frac{c_2^2}{2c} + \frac{b_1 c_1}{2a} - \frac{b_2 c_2}{2b} - \frac{b_3 c_3}{2c} = 0 \quad \dots\dots\dots(\mathfrak{A}),$$

$$c_{11} + a_{33} - \frac{c_1^2}{2c} - \frac{a_3^2}{2a} - \frac{c_1 a_1}{2a} + \frac{c_2 a_2}{2b} - \frac{c_3 a_3}{2c} = 0 \quad \dots\dots\dots(\mathfrak{B}),$$

$$a_{22} + b_{11} - \frac{a_2^2}{2a} - \frac{b_1^2}{2b} - \frac{a_1 b_1}{2a} - \frac{a_2 b_2}{2b} + \frac{a_3 b_3}{2c} = 0 \quad \dots\dots\dots(\mathfrak{C}),$$

$$a_{23} \qquad - \frac{a_2 a_3}{2a} - \frac{a_2 b_3}{2b} - \frac{a_3 c_2}{2c} \qquad = 0 \quad \dots\dots\dots(\mathfrak{F}),$$

$$b_{31} \qquad - \frac{b_1 a_3}{2a} - \frac{b_3 b_1}{2b} - \frac{b_3 c_1}{2c} \qquad = 0 \quad \dots\dots\dots(\mathfrak{G}),$$

$$c_{12} \qquad - \frac{c_1 a_2}{2a} - \frac{c_2 b_1}{2b} - \frac{c_1 c_2}{2c} \qquad = 0 \quad \dots\dots\dots(\mathfrak{H}).$$

These are in fact Lamé's equations, *Leçons sur les coordonnées curvilignes, etc.,* Paris (1859), pp. 76, 78, viz. the first of the equations (8), p. 76, is

$$\frac{d^2 H}{d\rho_1 d\rho_2} = \frac{1}{H_1} \frac{dH}{d\rho_1} \frac{dH_1}{d\rho_2} + \frac{1}{H_2} \frac{dH}{d\rho_2} \frac{dH_2}{d\rho_1},$$

which, in the notation of the present paper, is

$$(\sqrt{a})_{23} = \frac{1}{\sqrt{b}} (\sqrt{a})_2 (\sqrt{b})_3 + \frac{1}{\sqrt{c}} (\sqrt{a})_3 (\sqrt{c})_2.$$

Here

$$(\sqrt{a})_2 = \tfrac{1}{2} \frac{a_2}{\sqrt{a}},$$

$$(\sqrt{a})_{23} = \tfrac{1}{2} \left(\frac{a_2}{\sqrt{a}} \right)_3 = \tfrac{1}{2} \left\{ \frac{a_{23}}{\sqrt{a}} - \frac{\tfrac{1}{2} a_2 a_3}{a \sqrt{a}} \right\};$$

and the equation therefore is

$$\frac{a_{23}}{\sqrt{a}} - \frac{\tfrac{1}{2} a_2 a_3}{a \sqrt{a}} = \tfrac{1}{2} \frac{1}{\sqrt{b}} \frac{a_2}{\sqrt{a}} \frac{b_3}{\sqrt{b}} + \tfrac{1}{2} \frac{1}{\sqrt{c}} \frac{a_3}{\sqrt{a}} \frac{c_2}{\sqrt{c}},$$

which, multiplying by $\sqrt{a}$, gives the foregoing equation

$$a_{23} - \frac{a_2 a_3}{a} - \frac{a_2 b_3}{b} - \frac{a_3 c_2}{c} = 0.$$

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The first of the equations (9) p. 78 is

$$\frac{d}{d\rho_1}\left(\frac{1}{H_1}\frac{dH}{d\rho_1}\right) + \frac{d}{d\rho}\left(\frac{1}{H}\frac{dH_1}{d\rho}\right) + \frac{1}{H_2{}^2}\frac{dH}{d\rho_2}\frac{dH_1}{d\rho_2} = 0,$$

which, in the notation of the present paper, is

$$\left\{\frac{(\sqrt{a})_2}{\sqrt{b}}\right\}_2 + \left\{\frac{(\sqrt{b})_1}{\sqrt{a}}\right\}_1 + \frac{1}{c}(\sqrt{a})_3(\sqrt{b})_3 = 0.$$

This gives first

$$\frac{1}{2}\left(\frac{a_2}{\sqrt{a}\sqrt{b}}\right)_2 + \frac{1}{2}\left(\frac{b_1}{\sqrt{a}\sqrt{b}}\right)_1 + \frac{1}{4}\frac{a_3 b_3}{c\sqrt{a}\sqrt{b}} = 0,$$

and then

$$\frac{1}{2}\frac{a_{22}}{\sqrt{a}\sqrt{b}} - \frac{1}{4}\frac{a_2{}^2}{a\sqrt{a}\sqrt{b}} - \frac{1}{4}\frac{a_2 b_2}{b\sqrt{a}\sqrt{b}}$$
$$+ \frac{1}{2}\frac{b_{11}}{\sqrt{a}\sqrt{b}} - \frac{1}{4}\frac{b_1{}^2}{b\sqrt{a}\sqrt{b}} - \frac{1}{4}\frac{a_1 b_1}{a\sqrt{a}\sqrt{b}} \quad + \frac{1}{4}\frac{a_3 b_3}{c\sqrt{a}\sqrt{b}} = 0,$$

which, on multiplying by $\sqrt{a}\sqrt{b}$ and reducing, gives the foregoing equation

$$a_{22} + b_{11} - \frac{a_2{}^2}{a} - \frac{b_1{}^2}{b} - \frac{a_1 b_1}{a} - \frac{a_2 b_2}{b} + \frac{a_3 b_3}{c} = 0.$$