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THE PROBLEM OF POLYHEDRA.

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LET a, b, c, d, e, f, g, h, &c. represent the vertices of a polyhedron, then a face will be represented, e.g. by *abcde*, where the contiguous duads, viz. *ab*, *bc*, *cd*, *de*, *ea* are the edges of the face; and calling the face K, we may write

$$K = abcde. \tag{1}$$

It is to be noticed that the letters of a face-symbol may be taken forwards or backwards from any letter without altering the meaning of the symbol. Thus, *abcde*, *bcdea*, &c., *edcba*, &c. might any of them be taken to denote the face K. The diagonal of a face cannot be either an edge or a diagonal of any other face, i.e. a non-contiguous duad such as *ac* in a face-symbol K cannot be a duad, contiguous or non-contiguous, of any other face-symbol. But each edge of a face must be an edge of one and only one other face, i.e. each contiguous duad such as *ab* in the face-symbol K must be a contiguous duad of one and only one other face-symbol L. And moreover two faces cannot have more than a single edge in common, i.e. two face-symbols cannot contain more than a single contiguous duad, the same in each symbol.

The face K contains the edges *ab*, *ac*, i.e. the edge *ab* is contained in the face K; it will also be contained in one and only one other face, suppose L; this face will contain another edge through the vertex a, suppose the edge *af*, and so on, until we arrive at a face containing the edge *ae*; we have, for example,

$$K = eabcd,$$
$$L = baf\ldots,$$
$$M = fag\ldots,$$
$$N = gah\ldots,$$
$$P = hai\ldots,$$
$$Q = iae\ldots;$$

and we thence derive the vertex-symbol

$$a = KLMNPQ, \tag{2}$$

where the contiguous duads KL, LM, MN, NP, PQ, QK represent in order the edges through the vertex a. The remarks before made with respect to the face-symbols apply to the vertex-symbols. A non-contiguous duad such as KN of the vertex-symbol a cannot be a duad, contiguous or non-contiguous, of any other vertex-symbol; but each contiguous duad such as KL of the vertex-symbol a must be a contiguous duad of one and only one other vertex-symbol b. And the symbols of two vertices cannot contain more than one contiguous duad, the same in each symbol.

Any edge of the polyhedron admits of a double representation; it is the junction of two vertices, or the intersection of two faces. Thus ab and KL will represent the same edge, or we may write

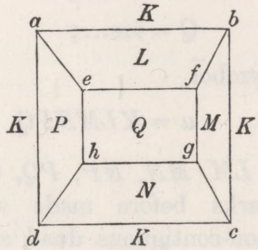
$$ab = KL. \tag{3}$$

It is to be remarked that in this system, to each equation $ab = KL$ there corresponds one and only one equation of the form $ae = KQ$, i.e. to an edge considered as drawn from a given vertex in a given face there corresponds one and only one other edge from the same vertex in the same face.

It has been shown how the system of face-symbols (1) leads to the system of vertex-symbols (2), and the system of edge symbols (3); and generally, any one of the three systems leads to the other two; and the three systems conjointly, or each system by itself, is a complete representation of the polyhedron. As an example, take the hexaedron: the three systems are:

(1)	(2)
$K = abcd,$	$a = LPK,$
$L = abfe,$	$b = LMK,$
$M = bfge,$	$c = MNK,$
$N = gcdh,$	$d = NPK,$
$P = dhea,$	$e = LPQ,$
$Q = hefg,$	$f = LMQ,$
	$g = MNQ,$
	$h = NPQ,$

$$ab = KL, \quad ae = PL, \quad ef = LQ, \quad (3)$$
$$bc = KM, \quad bf = LM, \quad fg = MQ,$$
$$cd = KN, \quad cg = MN, \quad gh = NQ,$$
$$de = KP, \quad dh = NP, \quad he = PQ.$$



Consider, now, two polyhedra having the same number of vertices and also the same number of faces. And let the vertices and faces of the first polyhedron taken in any order be represented by

$$abcde\ldots KLM\ldots,$$

and the vertices and faces of the second polyhedron taken in a certain order be represented by

$$a'b'c'd'e'\ldots K'L'M'\ldots;$$

then, forming the substitution symbol

$$a'b'c'd'e'\ldots K'L'M'\ldots abcde\ldots KLM\ldots,$$

which denotes that a' is to be written for a, b' for $b\ldots K'$ for K, &c., if operating with this upon the symbol system of the first polyhedron, we obtain the symbol system of the second polyhedron, the second polyhedron will be syntypic with the first. It should be noticed, that there may be several modes of arrangement of the vertices and faces of the second polyhedron, which will render it syntypic according to the foregoing definition with the first polyhedron, i.e. the second polyhedron may be syntypic in several different ways with the first polyhedron. This is, in fact, the same as saying that a polyhedron may be syntypic with itself in several different ways. Suppose, next, that the number of vertices of the second polyhedron is equal to the number of faces of the first polyhedron, and the number of faces of the second polyhedron is equal to the number of vertices of the first polyhedron; and let the vertices and faces of the first polyhedron in any order be represented by

$$abcde\ldots KLM\ldots,$$

and the faces and vertices of the second polyhedron in a certain order be represented by

$$A'B'C'D'E'\ldots k'l'm'\ldots.$$

Then, forming the substitution symbol

$$A'B'C'D'E' \ldots k'l'm' \ldots abcde \ldots KLM \ldots,$$

if, operating with this upon the symbol system of the first polyhedron, we obtain the symbol system of the second polyhedron, the second polyhedron is said to be polar-syntypic with the first; and, as in the case of syntypicism, this may happen in several different ways.

Lastly, if there be a polyhedron having the same number of vertices and faces, and if the vertices and faces in any order be represented by

$$abcd \ldots KLMN \ldots,$$

and the faces and vertices in a certain order be represented by

$$ABCD \ldots klmn \ldots;$$

then, forming the substitution symbol

$$ABCD \ldots klmn \ldots abcd \ldots KLMN \ldots,$$

if, operating with this upon the symbol system of the polyhedron, we reproduce such symbol system, i.e. in fact, if the polyhedron be polar-syntypic with itself, the polyhedron is said to be *autopolar;* and in accordance with a preceding remark, this may happen in several different ways. It is clear that the substitution symbol, operating on the symbol system of the vertices, must give the symbol system of the faces, and conversely; but operating on the symbol system of the edges, it must reproduce such symbol system of the edges: and this last condition will by itself suffice to make the polyhedron autopolar, i.e. the polyhedron will be autopolar if the substitution symbol, operating on the symbol system of the edges, reproduces such symbol system.