247.

ON THE ANALYTICAL FORMS CALLED TREES. SECOND PART.

[From the *Philosophical Magazine*, vol. XVIII. (1859), pp. 374—378. Continuation of **203**.]

THE following class of "trees" presented itself to me in some researches relating to functional symbols; viz., attending only to the terminal knots, the trees with one knot, two knots, three knots, and four knots respectively are shown in the figures 1, 2, 3 and 4:

Fig. 1

Fig. 2.

Fig. 3.

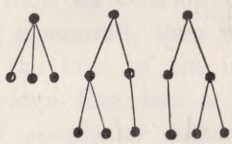
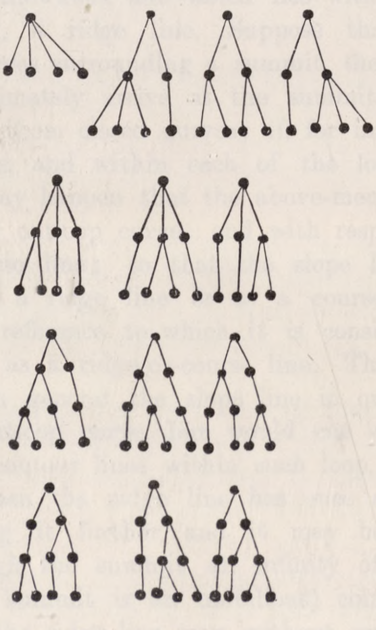


Fig. 4.



and similarly for any number of knots. The trees with four knots are formed first from those of one knot by attaching thereto in every possible way (one way only) four knotted branches; secondly, from those with two knots by attaching thereto in every possible way (three different ways) four knotted branches; and thirdly, from those with three knots by attaching thereto in every possible way (three different ways) four knotted branches,—the original knots of the trees of one knot and two and three knots, being no longer terminal knots, are disregarded. The total numbers of trees with one knot and with two and three knots being respectively 1, 1, 3; the total number of trees with four knots is $1.1+3.1+3.3=13$. And in general, if the number of trees with m knots is ϕm, then it is easy to see that we have

$$\phi m = \phi 1 + \frac{m-1}{1}\phi 2 + \frac{m-1 \, . \, m-2}{1 \, . \, 2}\phi 3 \ldots + \frac{m-1}{1}\phi(m-1);$$

or what is the same thing,

$$2\phi m = \phi 1 + \frac{m-1}{1}\phi 2 + \frac{m-1 \, . \, m-2}{1 \, . \, 2}\phi 3 \ldots + \frac{m-1}{1}\phi(m-1) + \phi m.$$

Hence if

$$u = \phi 1 + \frac{x}{1}\phi 2 + \frac{x^2}{1 \, . \, 2}\phi 2 + \ldots,$$

we obtain

$$\begin{aligned} e^x . u = & \quad \phi 1 \\ & + \; x \; (\phi 1 + \phi 2) \\ & + \frac{x^2}{1 \, . \, 2}(\phi 1 + 2\phi 2 + \phi 3) \\ & + \quad \&c. \\ = & \quad 2\phi 1 - 1 \\ & + \; x^2 . \, 2\phi 2 \\ & + \frac{x^2}{1 \, . \, 2} . \, 2\phi 3 \\ & + \quad \&c.; \end{aligned}$$

that is,

$$e^x u = 2u - 1,$$

and thence

$$u = \frac{1}{2-e^x},$$

which gives for ϕm the expression

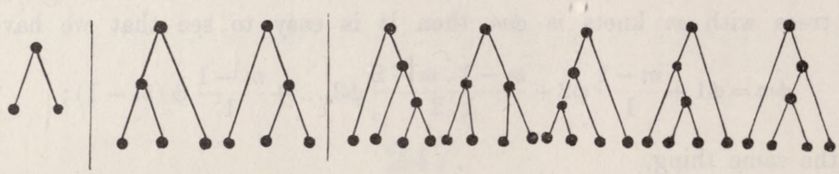
$$\phi m = 1 \, . \, 2 \, . \, 3 \ldots (m-1) \text{ coeff. } x^{m-1} \text{ in } \frac{1}{2-e^x};$$

and the value of ϕm might easily be obtained in an explicit form in terms of the differences of the powers of zero. The values of ϕm are, for

$$\begin{array}{rcccccccc} m = 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, \&c. \\ \phi m = 1, & 1, & 3, & 13, & 75, & 541, & 4683, & 47293. \end{array}$$

In the foregoing problem, the number of branches descending from a non-terminal knot is one, two, or more. But assume that the number of branches descending from a non-terminal knot is always two; so that attending, as before, only to the terminal knots, the trees with two knots, three knots, four knots respectively are shown in the figures, 5, 6, and 7.

Fig. 5. Fig. 6. Fig. 7.



This corresponds to the following problem in the theory of symbols; viz. if A, B, C, D, &c. are symbols capable of successive binary combinations, but do not satisfy the associative law, what is the number of the different significations of the ambiguous expressions ABC, $ABCD$, $ABCDE$, &c. respectively? For instance, AB has only one meaning; ABC may mean either $A.BC$ or $AB.C$. In like manner $ABCD$ may mean $A(B.CD)$, or $AB.CD$, or $(AB.C)D$, or $(A.BC)D$, or $A(BC.D)$; the numbers, 1, 2, 5 being those of the trees in the last three figures respectively; and similarly for any greater number of symbols.

Let ϕm be the required value corresponding to the number m; then we may in any manner whatever separate the number m into two parts m', m'', and then combining *inter se* the m' knots (or symbols) and the m'' knots (or symbols) respectively, ultimately combine the two combinations; hence a part of ϕm is $\phi m'.\phi m''$. The assumed definition of ϕm does not apply to the case $m=1$; but if we write $\phi 1=1$, then the foregoing consideration shows that we have

$$\begin{aligned}\phi m = \;& \phi 1 \phi(m-1)\\ & + \phi 2 \phi(m-2)\\ & \;\vdots\\ & + \phi(m-1)\phi 1;\end{aligned}$$

from which it is easy to calculate

$$\phi 1=1,\ \phi 2=1,\ \phi 3=2,\ \phi 4=5,\ \phi 5=14,\ \phi 6=42,\ \phi 7=132,\ \&c.$$

But to obtain the law, consider the generating function

$$u=\phi 1+\quad x\phi 2+\quad x^2\phi 3+\&c.;$$

we have

$$u^2=\phi 1\ \phi 1+x(\phi 1\ \phi 2+\phi 2\ \phi 1)+x^2(\phi 1\ \phi 3+\phi 2\ \phi 2+\phi 3\ \phi 1)+\&c.,$$

which is

$$=\phi 2+\quad x\phi 3+\quad x^2\phi 4+\&c.;$$

and we have therefore

$$xu^2 = u - 1,$$

and consequently

$$u = \frac{1 - \sqrt{1-4x}}{2x}.$$

But

$$\sqrt{1-4x} = 1 - \tfrac{1}{2}\,4x + \frac{\frac{1}{2}\,.\,-\frac{1}{2}}{1\,.\,2}(4x)^2 - \frac{\frac{1}{2}\,.\,-\frac{1}{2}\,.\,-\frac{3}{2}}{1\,.\,2\,.\,3}(4x)^3 + \text{\&c.}$$

$$= 1 - 2x - 2x^2 - 4x^3 - 10x^4 + \text{\&c.},$$

and therefore

$$u = \frac{1 - \sqrt{1-4x}}{2x} = 1 + 1x + 2x^2 + 5x^3 + \text{\&c.},$$

the series of coefficients 1, 1, 2, 5, &c. agreeing with the values already found. The expression for the general term is at once seen to be

$$\phi m = \frac{1\,.\,3\,.\,5\,\ldots\,2m-3}{1\,.\,2\,.\,3\,\ldots\,m}\,2^{m-1},$$

which is a remarkably simple form.

2, *Stone Buildings, W.C., June* 9, 1859.