244.

ON AN ANALYTICAL THEOREM RELATING TO THE DISTRIBUTION OF ELECTRICITY UPON SPHERICAL SURFACES.

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THERE is contained in Plana's "Mémoire sur la distribution de l'électricité à la surface de deux sphères conductrices complètement isolées" (*Mém. de Turin*, vol. VII. 1845), an identical relation which is remarkable, as well in itself as because by means of it the author corrects an error into which Poisson had fallen in his researches on the same subject. The development of a certain definite integral is obtained in the form (equation 165)

$$y = -2 \,.\, 3\,\frac{M_1 h}{b^2}\sin^2 \tfrac{1}{2}\theta + \frac{3 \,.\, 4 \,.\, 5}{1 \,.\, 2}\,\frac{M_2 h}{b^2}\sin^4 \tfrac{1}{2}\theta + \&c.$$

Poisson had in effect shown that $M_1 = 0$; and he thence inferred that, θ being small, the function in question $\propto \sin^4 \frac{1}{2}\theta$, or what is the same thing, $\propto (1-\cos\theta)^2$. In the former part of the memoir, Plana shows that this is not the true form of the development; the foregoing development must therefore be illusory; and Plana in fact shows, by a laborious induction carried as far as M_7, that all the coefficients M vanish identically. The identical equation $M_i = 0$, where i is any positive integer whatever, constitutes the analytical theorem above referred to. Plana's expression for the function M_i is as follows:

B_1, B_3, B_5, &c. denote Bernoulli's numbers as given by the equation

$$\frac{t}{e^t - 1} = 1 - \tfrac{1}{2}t + B_1\frac{t^2}{1 \,.\, 2} - B_3\frac{t^4}{1 \,.\, 2 \,.\, 3 \,.\, 4} + \&c.$$

($B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = \frac{1}{42}$, $B_7 = \frac{1}{30}$, &c. I have, in conformity with the usual practice written the equations so as to make these numbers all positive; with Plana they are alternately positive and negative). And in the equation 162, writing for k its value $\frac{1}{1+b}$, we have, λ being any positive integer,

$$(1+b)^\lambda G_\lambda = \frac{1}{\lambda+1} - \tfrac{1}{2}(1+b) \quad + \frac{\lambda . \lambda-1}{1.2} B_1 \frac{(1+b)^2}{\lambda-1}$$

$$- \frac{\lambda . \lambda-1 . \lambda-2 . \lambda-3}{1.2.3.4} B_3 \frac{(1+b)^4}{\lambda-3}$$

$$+ \frac{\lambda . \lambda-1 . \lambda-2 . \lambda-3 . \lambda-4 . \lambda-5}{1.2.3.4.5.6} B_5 \frac{(1+b)^6}{\lambda-5},$$

$$\pm \text{\&c.},$$

where the series is continued for so long as the factor in the denominator is positive. It should be observed that this factor really divides out, and that the rule just mentioned amounts to this, viz. that when λ is odd, the finite series on the right-hand side is to be continued to its last term; but when λ is even, the series is to be continued only to the last term but one. And G_λ being thus defined, the expression for M_i (see equation 164, in which I have written for k its value $\frac{1}{1+b}$) is

$$M_i = (1+b)^{i-1} \left\{ G_i + (i+2)\frac{1}{1}\frac{1+b}{b} G_{i+1} + (i+3)\frac{i}{1.2}\left(\frac{1+b}{b}\right)^2 G_{i+2} \right.$$

$$\left. + (i+4)\frac{i . i-1}{1.2.3}\left(\frac{1+b}{b}\right)^3 G_{i+3} + \text{\&c.} \right\},$$

where on the right-hand side the finite series is continued up to its last term, the value of which is obviously

$$(1+2i+1)\frac{1}{i+1}\left(\frac{1+b}{b}\right)^{i+1} G_{2i+1}, \text{ that is, } 2\left(\frac{1+b}{b}\right)^{i+1} G_{2i+1}.$$

But the form of this equation may be somewhat simplified. We in fact have

$$(1+b) M_i = \quad 1 \left\{ \frac{1}{i+1} - \tfrac{1}{2}(1+b) + B_1 \frac{i}{1.2}(1+b)^2 - B_3 \frac{i . i-1 . i-2}{1.2.3.4}(1+b)^4 + \text{\&c.} \right\}$$

$$+ \frac{i+2}{1}\frac{1}{b} \left\{ \frac{1}{i+2} - \tfrac{1}{2}(1+b) + B_1 \frac{i+1}{1.2}(1+b)^2 - B_3 \frac{i+1 . i . i-1}{1.2.3.4}(1+b)^4 + \text{\&c.} \right\}$$

$$+ \frac{i+3 . i}{1.2}\frac{1}{b^2} \left\{ \frac{1}{i+3} - \tfrac{1}{2}(1+b) + B_1 \frac{i+2}{1.2}(1+b)^2 - B_3 \frac{i+2 . i+1 . i}{1.2.3.4}(1+b)^4 + \text{\&c.} \right\},$$

which is at once changed into

$$(1+b) M_i =$$

$$\frac{1}{i+1}\left\{ 1 - \tfrac{1}{2}\frac{i+1}{1}(1+b) + B_1 \frac{i+1 . i}{1.2}(1+b)^2 - B_3 \frac{i+1 . i . i-1 . i-2}{1.2.3.4}(1+b)^4 + \text{\&c.} \right\}$$

$$+\tfrac{1}{1}\frac{1}{b}\left\{1-\tfrac{1}{2}\frac{i+2}{1}(1+b)+B_1\frac{i+2\,.\,i+1}{1\,.\,2}(1+b)^2-B_3\frac{i+2\,.\,i+1\,.\,i\,.\,i-1}{1\,.\,2\,.\,3\,.\,4}(1+b)^4+\&c.\right\}$$

$$+\frac{i}{1\,.\,2}\frac{1}{b^2}\left\{1-\tfrac{1}{2}\frac{i+3}{1}(1+b)+B_1\frac{i+3\,.\,i+2}{1\,.\,2}(1+b)^2-B_3\frac{i+3\,.\,i+2\,.\,i+1\,.\,i}{1\,.\,2\,.\,3\,.\,4}(1+b)^4+\&c.\right\}$$

$+ \&c.$;

or multiplying by $i+1$, and then putting $i-1$ in the place of i, we have

$$i\,(1+b)\,M_{i-1}=\Theta_i+\frac{i}{1}\frac{1}{b}\Theta_{i+1}+\frac{i\,.\,i-1}{1\,.\,2}\frac{1}{b^2}\Phi_{i+2}+\&c.=0,$$

where

$$\Theta_i=1-\tfrac{1}{2}\frac{i}{1}(1+b)+B_1\frac{i\,.\,i-1}{1\,.\,2}(1+b)^2-B_3\frac{i\,.\,i-1\,.\,i-2\,.\,i-3}{1\,.\,2\,.\,3\,.\,4}(1+b)^4\pm\&c.$$

In this last equation the finite series on the right-hand side is, when i is *even*, to be continued up to its last term, but when i is odd, then only up to the last term but one. The equation to be proved is

$$0=\Theta_i+\frac{i}{1}\frac{1}{b}\Theta_{i+1}+\frac{i\,.\,i-1}{1\,.\,2}\frac{1}{b^2}\Theta_{i+2}+\&c.,$$

where on the right-hand side the finite series is to be continued up to its last term: and the equation holds for any integer value of i which is >2. This is the simplest form of Plana's theorem.

We have

$$\Theta_i=i\,(1+b)^{i-1}\,G_{i-1}\,;$$

or writing this equation under the form $\Theta_i=i\,(1+b)\dfrac{G_{i-1}}{k^{i-2}}$, and comparing with Plana's developed expressions for $\dfrac{G_{i-1}}{k^{i-2}}$ (which are continued by him as far as G_{17}), we find

$$\begin{aligned}
\Theta_2 &= -\ b,\\
\Theta_3 &= -\tfrac{1}{2}\,b+\tfrac{1}{2}\,b^2,\\
\Theta_4 &= b^2,\\
\Theta_5 &= \tfrac{1}{6}\,b+\tfrac{2}{3}\,b^2-\tfrac{3}{2}\,b^3-\tfrac{1}{6}\,b^4,\\
\Theta_6 &= -\tfrac{1}{2}\,b^2-2\,b^3-\tfrac{1}{2}\,b^4,\\
\Theta_7 &= -\tfrac{1}{6}\,b-b^2-\tfrac{4}{3}\,b^3+\tfrac{4}{3}\,b^4+b^5+\tfrac{1}{6}\,b^6,\\
\Theta_8 &= \tfrac{2}{3}\,b^2+4\,b^3+\tfrac{23}{3}\,b^4+4\,b^5+\tfrac{2}{3}\,b^6,\\
\Theta_9 &= \tfrac{3}{10}\,b+\tfrac{12}{5}\,b^2+\tfrac{32}{5}\,b^3+\tfrac{24}{5}\,b^4-\tfrac{24}{5}\,b^5-\tfrac{32}{5}\,b^6-\tfrac{12}{5}\,b^7-\tfrac{3}{10}\,b^8,\\
\Theta_{10} &= -\tfrac{3}{2}\,b^2-12b^3-37b^4-54b^5-37\,b^6-12b^7-\tfrac{3}{2}\,b^8,
\end{aligned}$$

&c.,

which are of course the results obtained by developing the foregoing expression for Θ_i, in powers of b, and collecting the terms. The formulæ put in evidence a remarkable symmetry which does not exist in the original expression in powers of $1+b$.

It would be now easy to verify, for moderately small values of the suffix, the equations

$$\Theta_2 + \frac{2}{b}\Theta_3 + \frac{1}{b^2}\Theta_4 = 0,$$

$$\Theta_3 + \frac{3}{b}\Theta_4 + \frac{3}{b^2}\Theta_5 + \frac{1}{b^3}\Theta_6 = 0$$

&c.

This is, in fact, Plana's process, which, however, as the suffixes increase, becomes a very laborious one, and the law of the terms which destroy each other is not in anywise exhibited thereby.

I have succeeded in obtaining a complete demonstration, founded on Herschel's theorem for the development of a function of e^t, and the expression thereby given for Bernoulli's numbers. The theorem in question [See Herschel's *Collection of Examples in the Calculus of Finite Differences*, Cambridge, 1820, p. 70, where the theorem is given in the form $f\{(1+\Delta)^n\}\, 0^x = n^x . f(1+\Delta)\, 0^x$] is, that for any function of e^t which admits of development in positive integer powers of t,

$$f(e^t) = f(1+\Delta) e^{t\cdot 0},$$

where the right-hand side denotes the series the general term whereof is

$$\frac{t^n}{1.2.3\ldots n} f(1+\Delta)\, 0^n;$$

and $f(1+\Delta)$ is of course to be developed in powers of Δ, and the different terms Δ, Δ^2, Δ^3 &c., applied to the symbol 0^n (viz. $\Delta 0^n = 1^n - 0^n$, $\Delta^2 0^n = 2^n - 2.1^n + 0^n$, &c.). This gives

$$\frac{t}{e^t - 1} = \frac{\log(1+\Delta)}{\Delta} e^{t\cdot 0},$$

and comparing the development of the right-hand side with the development

$$1 - \tfrac{1}{2}t + B_1\frac{t^2}{1.2} - B_3\frac{t^4}{1.2.3.4} + \text{\&c.}$$

we find

$$\frac{\log(1+\Delta)}{\Delta}\, 0^0 = 1,$$

$$\frac{\log(1+\Delta)}{\Delta}\, 0^1 = -\tfrac{1}{2},$$

$$\frac{\log(1+\Delta)}{\Delta}\, 0^{2x} = (-)^{x+1} B_{2x-1},$$

$$\frac{\log(1+\Delta)}{\Delta}\, 0^{2x-1} = 0, \ (x > 1).$$

It is now easy to obtain the equation

$$\Theta_i = \frac{\log(1+\Delta)}{\Delta}\{(1+0\,(1+b))^i - (-0\,(1+b))^i\}.$$

In fact, the first two terms of the development of the expression on the right-hand side agree with those of the foregoing expression for Θ_i. For any even power $2x$ (except, when i is even, the power $2x=i$) the term is

$$\frac{[i]^{2x}}{[2x]^{2x}}(1+b)^{2x}\frac{\log(1+\Delta)}{\Delta}0^{2x},$$

which agrees; and when i is even, then for the power $2x=i$ there are two equal and opposite terms which destroy each other, and the whole term in Θ_i is, as it ought to be, zero. For any odd power $2x-1$, $(x>1)$, (including, when i is odd, the power $2x-1=i$), the term vanishes as containing an evanescent factor. The expression for Θ_i is thus shown to be true.

I write for shortness,

$$\Theta_i = \frac{\log(1+\Delta)}{\Delta}\{X^i - Y^i\},$$

where

$$X = 1+0\,(1+b),$$

$$Y = \quad -0\,(1+b).$$

Forming the expression for $i\,(1+b)\,M_{i-1}$,

$$= \Theta_i + \frac{i}{1}\frac{1}{b}\,\Theta_{i+1} + \frac{i\,.\,i-1}{1\,.\,2}\,\frac{1}{b^2}\,\Theta_{i+2} + \&c.,$$

this is

$$i\,(1+b)\,M_{i-1} = \frac{\log(1+\Delta)}{\Delta}\left\{\left(X\left(1+\frac{X}{b}\right)\right)^i - \left(Y\left(1+\frac{Y}{b}\right)\right)^i\right\},$$

and we have

$$X \quad = (1+0)\,(1+b) - b,$$

and therefore

$$1+\frac{X}{b} \quad = (1+0)\,\frac{1+b}{b},$$

$$\left(X\left(1+\frac{X}{b}\right)\right)^i = \left(\frac{1+b}{b}\right)^i((1+0)\,\{(1+0)\,(1+b)-b\})^i,$$

$$Y \qquad = -\,0\,(1+b),$$

$$1+\frac{Y}{b} = 1 - 0\,\frac{1+b}{b} = -\frac{1}{b}\{0\,(1+b)-b\},$$

$$\left(Y\left(1+\frac{Y}{b}\right)\right)^i = \left(\frac{1+b}{b}\right)^i(0\,\{0\,(1+b)-b\})^i.$$

We see that the expression for $\left(X\left(1+\frac{X}{b}\right)\right)^i$ is deduced from that of $\left(Y\left(1+\frac{Y}{b}\right)\right)^i$ by writing therein $1+0$ in the place of 0; we have therefore

$$\left(X\left(1+\frac{X}{b}\right)\right)^i = (1+\Delta)\left(Y\left(1+\frac{Y}{b}\right)\right)^i,$$

and consequently

$$\left(X\left(1+\frac{X}{b}\right)\right)^i - \left(Y\left(1+\frac{Y}{b}\right)\right)^i = \Delta\left(Y\left(1+\frac{Y}{b}\right)\right) = \left(\frac{1+b}{b}\right)\Delta\,(0\,\{0\,(1+b)-b\})^i;$$

whence also

$$i\,(1+b)\,M_{i-1} = \left(\frac{1+b}{b}\right)^i \log(1+\Delta)\,(0\,\{0\,(1+b)-b\})^i.$$

We have by the general theorem,

$$t = \log e^t = \log(1+\Delta)\,e^{t\cdot 0};$$

and consequently whenever $n \nless 2$,

$$\log(1+\Delta)\,0^n = 0.$$

But $i \nless 2$, and the function $(0\,\{0\,(1+b)-b\})^i$ contains only 0^i and the superior powers; it is therefore reduced to zero by the operation $\log(1+\Delta)$, and we have

$$i\,(1+b)\,M_{i-1} = \Theta_i + \frac{i}{1}\frac{1}{b}\,\Theta_{i+1} + \frac{i\,.\,i-1}{1\,.\,2}\frac{1}{b^2}\,\Theta_{i+2} + \&\text{c.} = 0;$$

and the theorem in question is thus proved. The foregoing expressions for Θ_2, Θ_3, &c. show that these functions all divide by b, and moreover that when i is even and greater than 2, then that Θ_i divides by b^2. The equation

$$\Theta_i = \frac{\log(1+\Delta)}{\Delta}\{(1+0\,(1+b))^i - (-0\,(1+b))^i\}$$

gives generally for the term in Θ_i involving b^α, the expression

$$\frac{[i]^\alpha}{[\alpha]^\alpha}\,b^\alpha\,\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{i-\alpha}\,0^\alpha - (-0)^i\};$$

and it is to be shown, first, that the coefficient vanishes for $\alpha=0$; and next, that when i is even and >2, the coefficient also vanishes for $\alpha=1$. Putting $\alpha=0$, the coefficient is

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^i - (-0)^i\},$$

which is equal to

$$\frac{\log(1+\Delta)}{\Delta}\{1+\Delta-(-)^i\,1\}\,0^i,$$

or to

$$\log(1+\Delta)\,0^i + \{1-(-)^i\,1\}\frac{\log(1+\Delta)}{\Delta}\,0^i,$$

where, since $i \nless 2$, the former term vanishes, as above remarked; and the latter term, when i is even, vanishes on account of the factor $1-(-)^i 1$; and when i is odd, on account of the other factor. Hence the coefficient vanishes for $\alpha = 0$.

Next, if i is even, and $\alpha = 1$, the coefficient becomes

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^{i-1}0 - 0^i\},$$

which, writing $(1+0)-1$ for 0, becomes

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^i - (1+0)^{i-1} - 0^i\},$$

which, since $(1+0)^i - 0^i = \Delta 0^i$, $(1+0)^{i-1} = (1+\Delta)\,0^{i-1}$, is equal to

$$\log(1+\Delta)\,0^i - \frac{(1+\Delta)\log(1+\Delta)}{\Delta}\,0^{i-1},$$

or since the first term vanishes, to

$$-\frac{(1+\Delta)\log(1+\Delta)}{\Delta}\,0^{i-1}.$$

But this function is to a numerical factor *près* the coefficient of t^{i-1} in $\frac{e^t \log e^t}{e^t - 1}$, or (what is the same thing) in $\frac{-t}{1-e^{-t}}$; and if in the expression for $\frac{t}{e^t-1}$ we write $-t$ in the place of t, we find

$$-\frac{t}{1-e^{-t}} = 1 + \tfrac{1}{2}t + B_1\frac{t^2}{1\,.\,2} - B_3\frac{t^4}{1\,.\,2\,.\,3\,.\,4} + \&c.,$$

so that, i being even and greater than 2, the function in question vanishes. Hence in the case under consideration the coefficient vanishes for $\alpha = 1$.

Writing β for $i - \alpha$, or assuming $\alpha + \beta = i$, the symmetry of the foregoing expressions for Θ_2, Θ_3, &c. shows that we ought to have

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta - (-0)^{\alpha+\beta}\} = \pm\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\beta 0^\alpha - (-0)^{\alpha+\beta}\},$$

where the upper or under sign is to be taken according as $\alpha + \beta$ is even or odd. Or separating the two cases, we find

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta - (1+0)^\beta 0^\alpha\} = 0, \quad \alpha+\beta \text{ even},$$

and

$$\frac{\log(1+\Delta)}{\Delta}\{(1+0)^\alpha 0^\beta + (1+0)^\beta 0^\alpha + 2\,.\,0^{\alpha+\beta}\} = 0, \quad \alpha+\beta \text{ odd}.$$

I have not attempted to verify *à posteriori* these elegant formulæ.

2, *Stone Buildings, W.C., June* 18, 1859.