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ON A CLASS OF DYNAMICAL PROBLEMS.

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THERE are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to (which might be designated as continuous-impact problems) are those in which the system is continually taking into connexion with itself particles of infinitesimal mass (i.e. of a mass containing the increment of time dt as a factor), so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion. For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of a table, the remainder of the chain being coiled or heaped up close to the edge of the table; the part hanging over constitutes the moving system, and in each element of time dt, the system takes into connexion with itself, and sets in motion with a finite velocity, an infinitesimal length ds of the chain; in fact, if v be the velocity of the part which hangs over, then the length vdt is set in motion with the finite velocity v. The general equation of dynamics applied to the case in hand will be

$$\Sigma\left\{\left(\frac{d^2x}{dt^2}-X\right)\delta x+\left(\frac{d^2y}{dt^2}-Y\right)\delta y+\left(\frac{d^2z}{dt^2}-Z\right)\delta z\right\}dm+\Sigma\left(\Delta u\delta\xi+\Delta v\delta\eta+\Delta w\delta\zeta\right)\frac{1}{dt}\,d\mu=0,$$

where the first term requires no explanation: in the second term ξ, η, ζ denote the coordinates at the time t of the particle $d\mu$ which then comes into connexion with the system; Δu, Δv, Δw are the finite increments of velocity (or, if the particle is originally at rest, then the finite velocities) of the particle $d\mu$ the instant that it has come into connexion with the system; $\delta\xi$, $\delta\eta$, $\delta\zeta$ are the virtual velocities of the same particle $d\mu$ considered as having come into connexion with and forming part of the system. The summation extends to the several particles or to the system of particles $d\mu$ which come into connexion with the system at the time t; of course, if there is only a single particle $d\mu$, the summatory sign Σ is to be omitted. The values of

Δu, Δv, Δw are $\frac{d\xi}{dt} - u$, $\frac{d\eta}{dt} - v$, $\frac{d\zeta}{dt} - w$, if by $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$ we understand the velocities of $d\mu$ parallel to the axes, after it has come into connexion with the system; but it is to be observed, that considering ξ, η, ζ as the coordinates of the particle $d\mu$ which is continually coming into connexion with the system, then if the problem were solved and ξ, η, ζ given as functions of t (and, when there is more than one particle $d\mu$, of the constant parameters which determine the particular particle), $\frac{d\xi}{dt}$, &c., in the sense just explained, cannot be obtained by simple differentiation from such values of ξ, &c.: in fact, ξ, η, ζ so given as functions of t, belong at the time t to one particle, and at the time $t + dt$ to the next particle, but what is wanted is the increment in the interval dt of the coordinates ξ, η, ζ of one and the same particle.

Suppose as usual that x, y, z, and in like manner that ξ, η, ζ are functions of a certain number of independent variables θ, ϕ, &c., and of the constant parameters which determine the particular particle dm or $d\mu$, of which x, y, z, or ξ, η, ζ are the coordinates; parameters, that is, which vary from one particle to another, but which are constant during the motion for one and the same particle. The summations are in fact of the nature of definite integrations in regard to these constant parameters, which therefore disappear altogether from the final results. The first term,

$$\Sigma \left\{ \left(\frac{d^2x}{dt^2} - X \right) \delta x + \left(\frac{d^2y}{dt^2} - Y \right) \delta y + \left(\frac{d^2z}{dt^2} - Z \right) \delta z \right\} dm,$$

may be reduced in the usual manner to the form

$$\Theta \delta\theta + \Phi \delta\phi + \ldots$$

where, writing as usual θ', ϕ', &c. for $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, &c., we have

$$\Theta = \frac{d}{dt} \frac{dT}{d\theta'} - \frac{dT}{d\theta} + \frac{dV}{d\theta},$$

$$\Phi = \frac{d}{dt} \frac{dT}{d\phi'} - \frac{dT}{d\phi} + \frac{dV}{d\phi}, \text{ \&c.,}$$

(this supposes that $Xdx + Ydy + Zdz$ is an exact differential); only it is to be observed that in the problems in hand, the mass of the system is variable, or what is the same thing, the variables θ, ϕ, &c., are introduced into T and V through the limiting conditions of the summation or definite integration, besides entering directly into T and V in the ordinary manner. And in forming the differential coefficients $\frac{d}{dt} \frac{dT}{d\theta'}$, $\frac{dT}{d\theta}$, $\frac{dV}{d\theta}$, &c., it is necessary to consider the variables θ, ϕ, &c., in so far as they enter through the limiting conditions *as exempt from differentiation*, so that the expressions just given for Θ, Φ, &c., are, in the case in hand, rather conventional representations than actual analytical values; this will be made clearer in the sequel by the consideration of the before-mentioned particular problem.

Considering next the second term, or

$$\Sigma\left\{\left(\frac{d\xi}{dt}-u\right)\delta\xi+\left(\frac{d\eta}{dt}-v\right)\delta\eta+\left(\frac{d\zeta}{dt}-w\right)\delta\zeta\right\}\frac{1}{dt}\,d\mu,$$

we have here

$$\begin{aligned}\delta\xi&=a\ \delta\theta+b\ \delta\phi+\dots,\\ \delta\eta&=a'\ \delta\theta+b'\ \delta\phi+\dots,\\ \delta\zeta&=a''\ \delta\theta+b''\ \delta\phi+\dots,\end{aligned}$$

where a, b, a', &c., are functions of the variables θ, ϕ, &c., and of the constant parameters which determine the particular particle $d\mu$. The virtual velocities or increments $\delta\theta$, $\delta\phi$, &c., are absolutely arbitrary, and if we replace them by $d\theta$, $d\phi$, &c., the actual increments of θ, ϕ, &c. in the interval dt during the motion, then $\delta\xi$, $\delta\eta$, $\delta\zeta$ will become $\frac{d\xi}{dt}dt$, $\frac{d\eta}{dt}dt$, $\frac{d\zeta}{dt}dt$, in the sense before attributed to $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$.

The particle $d\mu$ will contain dt as a factor, and the other factor will contain the differentials, or (as the case may be) products of differentials, of the constant parameters which determine the particular particle $d\mu$. We have thus the means of expressing the second line in the proper form; and if we write

$$\begin{aligned}\Sigma\,(a^2+a'^2+a''^2\ )\,d\mu&=\ Adt,\\ \Sigma\,(b^2\ +b'^2+b''^2\ )\,d\mu&=\ Bdt,\\ &\vdots\\ \Sigma\,(ab+a'b'+a''b'')\,d\mu&=\ Hdt,\\ &\vdots\\ \Sigma\,(au+a'v+a''w\ )\,d\mu&=-Pdt,\\ \Sigma\,(bu+b'v+b''w\ )\,d\mu&=-Qdt,\\ &\vdots\end{aligned}$$

then the required expression of the second line will be

$$(A\theta'+H\phi'\dots+P)\,\delta\theta+(H\theta'+B\phi'\dots+Q)\,\delta\phi+\dots,$$

which, if we put

$$\begin{aligned}K&=\tfrac{1}{2}\,(A\theta'^2+B\phi'^2+\dots+2H\theta'\phi'+\dots+2P\theta'+2Q\phi'+\dots),\\ &=\tfrac{1}{2}\,(A,\ B,\dots H,\dots P,\ Q,\dots)(\theta',\ \phi',\dots,\ 1)^2,\end{aligned}$$

may be more simply represented by

$$\frac{dK}{d\theta'}\,\delta\theta+\frac{dK}{d\phi'}\,\delta\phi+\dots;$$

only it is to be remarked that A, B, ... H, ... P, Q, ... will in general contain not only θ, ϕ, ..., but also the differential coefficients θ', ϕ', ..., and that in forming the differential cœfficients $\frac{dK}{d\theta'}$, $\frac{dK}{d\phi'}$, &c., the quantities θ', ϕ', ..., in so far as they enter into K, not explicitly, but through the coefficients A, &c., must be considered *as exempt from differentiation*, so that the preceding expression for the second line by means of the function K is rather a conventional representation than an actual analytical value.

Uniting the two terms, and equating to zero the coefficients of $\delta\theta$, $\delta\phi$, &c., we obtain finally the equations of motion in the form

$$\frac{d}{dt}\frac{dT}{d\theta'}-\frac{dT}{d\theta}+\frac{dV}{d\theta}+\frac{dK}{d\theta'}=0,$$

$$\frac{d}{dt}\frac{dT}{d\phi'}-\frac{dT}{d\phi}+\frac{dV}{d\phi}+\frac{dK}{d\phi'}=0,$$

$$\vdots$$

where the several symbols are to be taken in the significations before explained.

In the particular problem, let z be measured vertically downwards from the plane of the table, then $Z=g$, and repeating for the particular case the investigation *ab initio*, the general equation of motion is

$$\Sigma\left(\frac{d^2z}{dt^2}-g\right)\delta z\, dm+\frac{d\zeta}{dt}\,\delta\zeta\,\frac{1}{dt}\,d\mu=0.$$

Let s be the length in motion, or, what is the same thing, the z coordinate of the lower extremity; and suppose also that the mass of a unit of length is taken equal to unity, we have $\delta z=\delta s$, $\frac{d^2z}{dt^2}=\frac{d^2s}{dt^2}$, $dm=dz$, and the summation or integration with respect to z is from $z=0$ to $z=s$, whence

$$\Sigma\left(\frac{d^2z}{dt^2}-g\right)\delta z\, dm=\left(\frac{d^2s}{dt^2}-g\right)\delta s\,\Sigma dz=\left(\frac{d^2s}{dt^2}-g\right)s\delta s\,;$$

which is of the form

$$\left(\frac{d}{dt}\frac{dT}{ds'}-\frac{dT}{ds}+\frac{dV}{ds}\right)\delta s,$$

if

$$T=\tfrac{1}{2}s'^2\,.\,\bar{s},\quad V=-gs\bar{s},$$

where the bar is used to denote exemption from differentiation, but ultimately $\bar{s}$ is to be replaced by s. Considering now the second term, here $\zeta=0$, but $\delta\zeta=\delta s$, and thence $\frac{d\zeta}{dt}=s'$. Moreover, $d\mu=s'dt$, and thence finally the second term is s'^2, which is of the form $\frac{dK}{ds'}$, if

$$K=\tfrac{1}{2}\,\overline{s'}\,.\,s'^2,$$

the bar having the same signification as before, but after the differentiation $\overline{s'}=s'$. The resulting equation is

$$\left(\frac{d^2s}{dt^2}-g\right)s+\left(\frac{ds}{dt}\right)^2=0,$$

which may be written in the form

$$s\frac{ds}{dt}\,d\left(s\frac{ds}{dt}\right)=gs^2ds,$$

and the first integral is therefore

$$\frac{sds}{\sqrt{s^3-a^3}}=\sqrt{\frac{2g}{3}}\,dt,$$

where a is the length hanging over at the commencement of the motion. If $a=0$, then the equation is

$$\frac{ds}{\sqrt{s}}=\sqrt{\frac{2g}{3}}\,dt,$$

and integrating from $t=0$, $2\sqrt{s}=\sqrt{\dfrac{2g}{3}}\,t$, or finally $s=\frac{1}{6}gt^2$, so that the motion is the same as that of a body falling under the influence of a constant force $\frac{1}{3}g$. It is perhaps worth noticing that the differential equation may be obtained as follows:—We have, in the first instance, a mass s moving with a velocity s', and after the particle $ds\,(=s'dt)$ has been set in motion, a mass $s+s'dt$ moving say with a velocity $s'+\delta s'$, whence neglecting for the moment the effect of gravity on the mass s, the momentum of the mass in motion will be constant, or we shall have

$$ss'=(s+s'dt)(s'+\delta s')=ss'+s'^2dt+s\delta s',$$

and therefore $s\delta s'=-s'^2dt$. Hence, adding on the right-hand side the term $gsdt$ arising from gravity, and substituting $\dfrac{d^2s}{dt^2}dt$ for $\delta s'$, we have the equation, $s\dfrac{d^2s}{dt^2}=gs-\left(\dfrac{ds}{dt}\right)^2$, as before.