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ON PROFESSOR MAC CULLAGH'S THEOREM OF THE POLAR PLANE.

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A RAY of polarized light, incident on the surface of an extraordinary medium, may give rise to a reflected ray and a single refracted ray; but this will be the case only for a particular position, or positions, of the plane of polarization of the incident ray. According to Professor Mac Cullagh's theory, the plane of polarization, and the relative vibrations of the three rays, are deduced from two assumed principles, which may be referred to as

1°. The principle of equivalent vibrations.

2°. The principle of equivalent moments.

And from these principles are deduced

3°. The principle of vis viva.

4°. The theorem of the polar plane.

The directions of the vibrations are completely determined by means of 4°, the theorem of the polar plane; and the relative magnitudes are then given by 1°, the principle of equivalent vibrations. The other principles, viz.:—2°, the principle of equivalent moments, and 3°, the principle of vis viva, must therefore follow as mere geometrical consequences from the first-mentioned two principles, or theorems; and I have found that the deduction depends immediately upon the following two theorems in spherical trigonometry.

Suppose (Fig. 1) that $RR'R''$ is a spherical triangle, and let W be any point in the base RR'', and N be the central point of the base; then joining WR' and

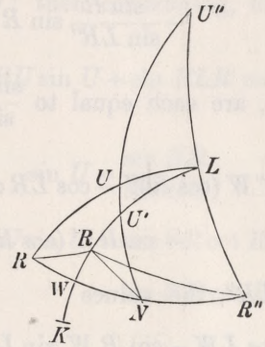

Fig. 1.

producing this arc (in the direction from W to R') to a point L, such that

$$\cot WL = \frac{\sin^2 NW}{\sin WR \sin WR'} \tan WR',$$

and joining NR', then

THEOREM I.

$$\frac{\sin^2 R''LR' - \sin^2 RLR'}{\sin^2 R''LR} = \frac{\sin NW \cos NR'}{\cos WR' \sin NR \cos NR'};$$

and if we suppose also, that an arc through N, perpendicular to the base RR'', cuts LR, LR', and LR'' produced in the points U, U', U'', then

THEOREM II.

$$\sin R''LR' \cos RU \sin NUR + \sin RLR' \cos R''U'' \sin NU''R''$$

$$= \sin R''LR \frac{\cos NW}{\cos WR' \sin NR} \cos R'U' \sin NU'R'.$$

The present memoir contains the proof of the two theorems, and the application of them to the optical theory.

To prove the first theorem, I write for shortness R, R'', W to denote the angles LRR'', $LR''R$, NWR', respectively; we have then,

$$\frac{\sin^2 R''LR' - \sin^2 RLR'}{\sin^2 R''LR} = \frac{\sin (R''LR' - RLR')}{\sin R''LR}$$

$$= \frac{1}{\sin R''LR} \{\sin R''LR' \cos RLR' - \sin RLR' \cos R''LR'\}$$

$$= \frac{1}{\sin R''LR} \left\{ \frac{\sin R''W \sin R''}{\sin LW} \cdot \frac{\cos RW - \cos LR \cos LW}{\sin LR \sin LW} \right.$$

$$\left. - \frac{\sin RW \sin R}{\sin LW} \cdot \frac{\cos R''W - \cos LR'' \cos LW}{\sin LR'' \sin LW} \right\}$$

$$= \frac{1}{\sin R''LR \sin^2 LW} \left\{ \frac{\sin R''}{\sin LR} \sin R''W (\cos RW - \cos LR \cos LW) \right.$$
$$\left. - \frac{\sin R}{\sin LR''} \sin RW (\cos R''W - \cos LR'' \cos LW) \right\}.$$

Observing that $\frac{\sin R''}{\sin LR}$, $\frac{\sin R}{\sin LR''}$, are each equal to $\frac{\sin R''LR}{\sin R''R}$, this becomes

$$= \frac{1}{\sin R''R \sin^2 LW} \{\sin R''W (\cos RW - \cos LR \cos LW)$$
$$- \sin RW (\cos R''W - \cos LR'' \cos LW)\};$$

and, substituting for $\cos LR$, $\cos LR''$, the values

$$\cos R\,W \cos LW - \sin R\,W \sin LW \cos W,$$
$$\cos R''W \cos LW + \sin R''W \sin LW \cos W,$$

the foregoing expression becomes

$$= \frac{1}{\sin R''R \sin^2 LW} \times \{\sin R''W (\cos R\,W \sin^2 LW + \sin R\,W \sin LW \cos LW \cos W)$$
$$- \sin RW\ (\cos R''W \sin^2 LW - \sin R''W \sin LW \cos LW \cos W)\},$$

$$= \frac{1}{\sin R''R} \{\sin R''W \cos RW - \sin RW \cos R''W + 2 \cot LW \sin RW \sin R''W \cos W\}$$

$$= \frac{1}{\sin R''R} \{\sin (R''W - RW) + 2 \cot LW \sin RW \sin R''W \cos W)\};$$

and, putting $R''W - RW = 2NW$, and substituting also for $\cot WL$ its value, which gives $\cot LW \sin RW \sin R''W = \sin^2 NW \tan WR'$, the expression becomes

$$= \frac{1}{\sin R''R} \{\sin 2NW + 2 \sin^2 NW \tan WR' \cos W\};$$

but we have

$$\cos W = \frac{\cos NR' - \cos NW \cos WR'}{\sin NW \sin WR};$$

and therefore

$$2 \sin^2 NW \tan WR' \cos W = 2 \frac{\sin NW}{\cos WR'} \cos NR' - \sin 2NW;$$

thus the expression becomes

$$= \frac{1}{\sin R''R}\, 2 \frac{\sin NW}{\cos WR'} \cos NR';$$

and $\sin R''R = \sin 2NR = 2 \sin NR \cos NR$, so that finally the expression becomes

$$= \frac{\sin NW \cos NR'}{\cos WR' \sin NR \cos NR},$$

which proves the theorem.

To prove the second theorem, take as before R, R'', W, to denote the angles LRR'', $LR''R$, NWR', respectively; and moreover, U, U', U'' to denote the angles NUR, $NU'R'$, $NU''R''$, respectively; then considering, first, the function on the left-hand side, viz.:

$$\sin R''LR' \cos RU \sin U + \sin RLR' \cos R''U'' \sin U'',$$

we have

$$\sin U = \frac{\sin NR}{\sin RU},$$

$$\cos RU \sin U = \sin NR \cot RU$$

$$= \sin NR \cos R \cot NR = \cos R \cos NR,$$

and, in like manner,

$$\sin U'' = \frac{\sin NR''}{\sin R''U''},$$

$$\cos R''U'' \sin U'' = \sin NR'' \cot R''U'' = \sin NR'' \cos R'' \cot NR''$$

$$= \cos R'' \cos NR'' = \cos R'' \cos NR;$$

the expression thus becomes

$$= \cos NR \{\sin R''LR' \cos R + \sin RLR' \cos R''\},$$

which is

$$= \cos NR \left\{\frac{\sin R''W \sin W}{\sin R''L} \cdot \frac{\cos WL - \cos RW \cos RL}{\sin RW \sin RL} - \frac{\sin RW \sin W}{\sin RL} \cdot \frac{\cos WL - \cos R''W \cos R''L}{\sin R''W \sin R''L}\right\},$$

or, substituting for $\cos RL$, $\cos R''L$ the values

$$\cos R\,W \cos WL - \sin R\,W \sin WL \cos W,$$

$$\cos R''W \cos WL + \sin R''W \sin WL \cos W,$$

the expression becomes

$$\frac{\cos NR \sin W}{\sin RL \sin R''L}\left\{\frac{\sin R''W}{\sin RW}(\cos WL \sin^2 R\,W + \sin WL \sin R\,W \cos R\,W \cos W) + \frac{\sin RW}{\sin R''W}(\cos WL \sin^2 R''W - \sin WL \sin R''W \cos R''W \cos W)\right\}$$

$$= \frac{\cos NR \sin W}{\sin RL \sin R''L}\{2 \cos WL \sin RW \sin R''W + \sin WL \sin (R''W - RW) \cos W\}$$

$$= \frac{\cos NR \sin W \sin WL}{\sin RL \sin R''L}\{2 \cot WL \sin RW \sin R''W + \sin (R''W - RW) \cos W\}.$$

Hence, putting for cot WL its value, which gives

$$\cot WL \sin RW \sin R''W = \sin^2 NW \tan WR',$$

and putting also

$$\sin (R''W - RW) = \sin 2NW = 2 \sin NW \cos NW,$$

the expression becomes

$$= \frac{2 \cos NR \sin W \sin WL \sin^2 NW}{\sin RL \sin R''L} (\tan WR' + \cot NW \cos W).$$

The right-hand side of the equation to be proved is

$$\sin R''LR \frac{\sin NW}{\cos WR' \sin NR} \cos R'U' \sin U',$$

and we have

$$\sin R''LR = \frac{\sin RR'' \sin R}{\sin R''L}, \quad \sin R = \frac{\sin WL \sin W}{\sin RL},$$

and consequently

$$\sin R''LR = \frac{\sin RR'' \sin WL \sin W}{\sin RL \sin R''L} = \frac{2 \sin NR \cos NR \sin NL \sin W}{\sin RL \sin R''L},$$

or the expression is

$$= \frac{2 \cos NR \sin WL \sin W}{\sin RL \sin R''L} \cdot \frac{\sin NW}{\cos WR'} \cos R'U' \sin U'.$$

But we have

$$\sin U' = \frac{\sin NW}{\sin W'U'};$$

and therefore

$$\frac{\sin NW}{\cos WR'} \cos R'U' \sin U' = \sin^2 NW \frac{\cos R'U'}{\cos WR' \sin WU'}$$

$$= \sin^2 NW \frac{\cos (WU' - WR')}{\cos WR' \sin WU'}$$

$$= \sin^2 NW (\tan WR' + \cot WU').$$

Moreover we have $\cot WU' = \cot NW \cos W$, and the expression thus becomes

$$= \frac{2 \cos NR \sin W \sin WL \sin^2 NW}{\sin RL \sin R''L} (\tan WR' + \cot NW \cos W);$$

which is the expression previously found as the value of the left-hand side of the equation, and the theorem is therefore proved.

It is obvious that the point L might have been constructed by taking on $R'W$, produced in the direction from R' to W, a point K such that

$$\tan KW = \frac{\sin^2 NW}{\sin RW \sin R''W} \tan WR',$$

and then taking the arc KL in the reverse direction equal to 90°.

Passing now to the optical problem, it will be recollected that in Mac Cullagh's theory the direction of vibration in an extraordinary medium is perpendicular to the plane of the ray and wave normal, and that the polar plane of a refracted ray is by definition a plane through the point of incidence parallel to the direction of vibration, and also parallel to a line joining the extremity of the ray with the corresponding point on the *Index surface*,—the last-mentioned surface being the polar reciprocal of the refracted wave-surface, taken with respect to the reflected wave-surface, or wave-sphere, contemporaneously generated. We have to consider a ray of polarized light incident on the surface of an extraordinary medium, and giving rise to a reflected ray and a single refracted ray. Let the incident ray and the reflected ray be respectively produced within the medium, and let the three rays, viz., the incident ray produced, the refracted ray, and the reflected ray produced, be represented in direction (see Fig. 2) by AR, AR' and AR''; and take $AR = AR'' = 1$ as the radius of the wave-sphere and AR' as the radius of the wave-surface, corresponding at a given instant

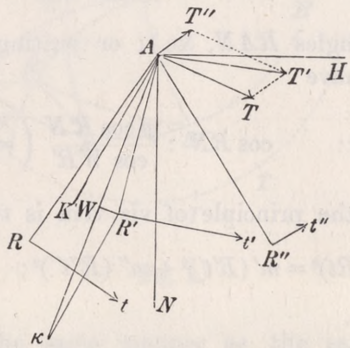

Fig. 2.

of time to the first or ordinary medium and the extraordinary medium respectively. Take also AW as the perpendicular on the tangent plane of the wave-surface at R', or 'wave-normal,' corresponding to the refracted ray AR'; and let AN represent the normal to the plane of separation of the two media, and AH the intersection of the last-mentioned plane with the plane of incidence. The lines AR, AR'', AW, AN, AH are of course all of them in the plane of incidence, the line AN bisects the angle made by the lines AR, AR'', and the lines AN, AH are at right angles to each other. The length of the wave-normal AW is given by the equation $AR \sin NAW = AW \sin NAR$, or putting, as above, $AR = 1$, and representing the two angles at A by NW, NR respectively, then, if p denote the length of the wave-normal, we have $\sin NW = p \sin NR$.

Take κ the pole of the tangent plane of the wave-surface at R' (or, what is the same thing, the image of the point W), in respect of the sphere radius AR, then κ will be the point on the index-surface corresponding to the point R' of the wave-surface; and let AK be drawn through the point A parallel to $R'\kappa$. Take AT' perpendicular to the plane WAR' (or, what is the same thing, the plane KAR') as the direction of the refracted vibration, the plane KAT' will be the polar plane; and by 4°, the theorem of the polar plane, the directions of the incident and reflected vibrations are given as the intersections of the polar plane with the wave-fronts or planes through A normal to the directions of the incident and reflected rays respectively; these intersections are represented in the figure by AT and AT''. The relative magnitudes of the vibrations are then determined by 2°, the principle of equivalent vibrations, viz., considering these vibrations as forces acting in the given directions AT', AT, AT'' respectively, the refracted vibration will be the resultant of the incident and reflected vibrations: the terminated lines AT', AT, AT'' in the figure are taken to represent in direction and magnitude the vibrations corresponding to the refracted ray and to the incident and reflected rays respectively, and the lines $R't'$, Rt, $R''t''$ are drawn through the extremities R', R, R'' of the three rays equal and parallel to AT', AT, and AT'' respectively. Let m', m, m'' denote the masses of ether set in motion by the three rays respectively, then, according to Mac Cullagh's hypothesis of equal densities, we have

$$m = m'' : m' :: AR\cos RN : \frac{AW\cos R'N}{\cos WR'},$$

(where RN, &c., denote the angles RAN, &c.); or writing as before, $AR=1$, $AW=p$, where $\sin NW = p\sin RN$, we have

$$m = m'' : m' :: \quad \cos RN : \frac{p\cos R'N}{\cos WR'}\left(=\frac{\sin NW\cos R'N}{\cos WR'\sin RN}\right).$$

This being premised, then, 3°, the principle of vis viva is that

$$m\,(Rt)^2 = m'\,(R't')^2 + m''\,(R''t'')^2;$$

or, what is the same thing,

$$\frac{Rt^2 - R''t''^2}{R't'^2} = \frac{m'}{m} = \frac{\sin NW\cos R'N}{\cos WR'\sin RN\cos RN};$$

and 2°, the principle of equivalent moments, is that the moment of $R't'$ round the axis AH, is equal to the sum of the moments of Rt and $R''t''$ round the same axis. It only remains to show that these two properties are in fact contained in the Theorems I. and II.

The point κ is the image of W in a sphere, radius unity. Hence, $A\kappa = \frac{1}{p}\,\kappa W = \frac{1}{p} - p$, and therefore

$$\tan W\kappa R' = \frac{p^2\tan WR'}{1-p^2} = \tan KW,$$

but we have, as before, $\sin NW = p \sin RN$, and consequently,

$$\tan KW = \frac{\sin^2 NW \tan WR'}{\sin^2 RN - \sin^2 RW}$$

$$= \frac{\sin^2 NW}{\sin RW \sin R''W} \tan WR'.$$

Suppose now that the points R, R', R'', W, N, H, K, of Fig. 2, are all of them projected by radii through the centre A upon a sphere, radius unity (see Fig. 3, where the several points are represented by the same letters as in Fig. 2); and complete Fig. 3 by connecting the different points in question by arcs of great circles, and by producing KW (in the direction from K to W) to a point L, such that $KL = 90°$, and by joining LR, LR'', and drawing the arc $NU'UU''$ at right angles to $R''R$ (or, what is the same thing, with the pole H) meeting LR', LR, and LR'' produced, in the points U', U, U'' respectively. By what has preceded, the points K, L of Fig. 3

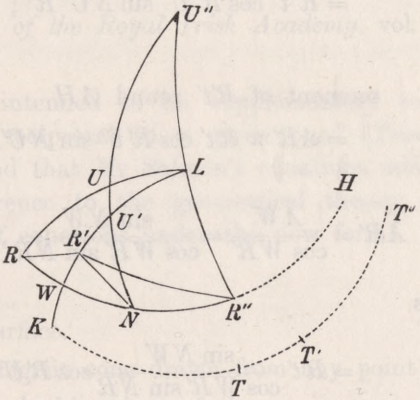

Fig. 3.

are constructed precisely in the same manner as the same points in Fig, 1, and in fact Fig. 3 is nothing else than Fig. 1 with some additional lines and points. The condition employed to determine the magnitude of the vibrations Rt, $R't'$, $R''t''$, gives that these vibrations are as

$$\sin T'T'' : \sin TT'' : \sin TT',$$

or, observing that LR, LR', LR'' are the great circles whose poles are T, T', T'' respectively, these vibrations are as

$$\sin R''LR' : \sin RLR'' : \sin RLR';$$

and, substituting these values, the equation given by the principle of vis viva becomes identical with that of Theorem I.

Proceeding to the condition given by the principle of equivalent moments, we have

$$\text{moment of } Rt \text{ round } AH$$

$$= Rt \times AR \times \cos [AR, \perp \text{dist.} (Rt, AH)] \times \sin (Rt, AH);$$

and in Fig. 3, observing that the radius through U is parallel to the perpendicular distance of (Rt, AH) (for LR has the pole T, and NU the pole H) then

$$\cos [AR, \perp \text{dist.} (Rt, AH)] = \cos RU,$$

$$\sin (Rt, AH) = \sin TH,$$

or, since T and H are the poles of LR and NW respectively, $TH = \angle NUR$, and, putting $AR = 1$, the moment is

$$= Rt \cos RU \sin NUR.$$

Similarly,

$$\text{moment of } R''t'' \text{ round } AH$$

$$= R''t'' \cos R''U'' \sin NU''R'';$$

and for the refracted ray,

$$\text{moment of } R't' \text{ round } AH$$

$$= AR' \times R't' \cos R'U' \sin NU'R'.$$

But we have

$$AR' = \frac{AW}{\cos WR'} = \frac{\sin NW}{\cos WR' \sin NR},$$

and therefore the moment is

$$= R't' \frac{\sin NW}{\cos WR' \sin NR} \cos R'U' \sin NU'R'.$$

Hence the vibrations Rt, $R''t''$, $R't'$, as before, are as

$$\sin R''LR' : \sin RLR' : \sin RLR'',$$

and thus the equation given by the principle of equivalent moments is precisely that of Theorem II.