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ON THE CUSP OF THE SECOND KIND OR NODECUSP.

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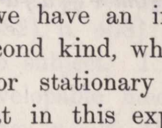
THE so-called cusp of the second kind or ramphoid cusp, is not an ordinary singularity of plane curves, but it is a singularity of a higher order. It is however particularly considered in Plücker's *Theorie der Analytischen Curven,* 1839; and it is there, not only in the analytical discussion of the singularities of plane curves, but in the author's theory of the generation of a curve, considered as described and enveloped by a point moving along a line which at the same time rotates round the point; when the motion along the line vanishes, we have a cusp; when the motion round the point vanishes, we have an inflexion; when the two motions vanish together, we have a cusp of the second kind, which thus presents itself as a singularity uniting the characters of a cusp or stationary point, and an inflexion or stationary tangent: (I remark in passing that in this explanation it is not clear what is the independent variable wherewith the motions are compared). But there is another point of view from which the singularity in question may be considered, viz., it may be regarded as a singularity arising from the union and amalgamation of a cusp, and a double point or node; in fact, in the figure, which represents a curve having a cusp



and also a node, we have only to imagine the node approaching nearer and nearer to and ultimately coinciding with the cusp, and it will be at once seen that the point will become a cusp of the second kind; or as it might properly, with reference to this generation of it, be termed, a "nodecusp." It is to be noticed that in the point-theory of curves, there is between the cusp and the nodecusp the intermediate singularity of the tacnode, which arises from the union and amalgamation of two nodes, and possesses the character of a cusp.

I return to the nodecusp; taking the point in question as the origin, and the tangent for the axis of x, the equation will be a specialised form of the equation

$$\tfrac{1}{2}y^2 + \tfrac{1}{6}(a, b, c, d \between x, y)^3 + \tfrac{1}{24}(a', b', c', d', e' \between x, y)^4 + \&c. = 0,$$

which belongs to the case of a cusp, viz. (see Plücker, p. 165) the conditions satisfied by the special form are $a = 0$, $a' = 3b^2$, or the equation is

$$\tfrac{1}{2}(y + \tfrac{1}{2}bx^2)^2 + \tfrac{1}{6}y^2(3cx + dy) + \tfrac{1}{24}y(4b'x^3 + 6c'x^2y + 4d'xy^2 + e'y^3) + \&c. = 0,$$

which is most easily verified, by observing that (this being so) the expansion of y in terms of x will be of the form

$$y = -\tfrac{1}{2}bx^2 + Ax^{\frac{5}{2}} + \&c.$$

It is now to be shown how the foregoing conditions $a = 0$, $a' = 3b^2$, are obtained by assuming that the curve has, besides the cusp, a node which ultimately coincides with the cusp. Let (α, β) be the coordinates of the node; we must have

$$\begin{aligned}
\tfrac{1}{2}\beta^2 + \tfrac{1}{6}(a, b, c, d \between \alpha, \beta)^3 + \tfrac{1}{24}(a', b', c', d', e' \between \alpha, \beta)^4 + \&c. &= 0,\\
\tfrac{1}{2}(a, b, c \between \alpha, \beta)^2 + \tfrac{1}{6}(a', b', c', d' \between \alpha, \beta)^3 + \&c. &= 0,\\
\beta + \tfrac{1}{2}(b, c, d \between \alpha, \beta)^2 + \tfrac{1}{6}(b', c', d', e' \between \alpha, \beta)^3 + \&c. &= 0.
\end{aligned}$$

Assume $\beta = m\alpha^2$, and then let α vanish; the equations become in the first instance

$$\begin{aligned}
\tfrac{1}{6}a\alpha^3 + (\tfrac{1}{2}m^2 + \tfrac{1}{2}bm + \tfrac{1}{24}a')\,\alpha^4 + \&c. &= 0,\\
\tfrac{1}{2}a\alpha^2 + \&c. &= 0,\\
(m + \tfrac{1}{2}b)\,\alpha^2 + \&c. &= 0,
\end{aligned}$$

the second and third equation give $a = 0$, $m + \tfrac{1}{2}b = 0$, and the first equation then gives

$$\tfrac{1}{2}m^2 + \tfrac{1}{2}bm + \tfrac{1}{24}a' = 0;$$

or substituting for m its value $= -\tfrac{1}{2}b$, this is $a' = 3b^2$, or the required conditions are $a = 0$, $a' = 3b^2$, *ut suprà*. The single condition $a = 0$ corresponds to the case of the tacnode.

2, *Stone Buildings, W.C., September* 17, 1862.