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ANALYTICAL THEOREM RELATING TO THE SECTIONS OF A QUADRIC SURFACE.

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THE four sections $x=0$, $y=0$, $z=0$, $w=0$ of the quadric surface

$$ax^2+by^2+6xy\sqrt{ab}-cz^2-dw^2=0$$

are each of them touched by each of the four sections

$$x\sqrt{2a}+y\sqrt{2b}\pm z\sqrt{c}\pm w\sqrt{d}=0;$$

where it is to be noticed that the radicals $\sqrt{2a}$, $\sqrt{2b}$ are such that their product is $=2\sqrt{ab}$ if $\sqrt{ab}$ be the radical contained in the equation of the surface. There is of course no loss of generality in attributing a definite sign to the radical $\sqrt{2a}$; but upon this being done, the sign of the radical $\sqrt{2b}$ is determined, whereas the signs of $\sqrt{c}$ and $\sqrt{d}$ are severally arbitrary. We may if we please write the equation of any one of the last-mentioned sections in the form

$$x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0,$$

it being understood that the radicals $\sqrt{2a}$, $\sqrt{2b}$ have each a determinate sign, but that the signs of $\sqrt{c}$ and $\sqrt{d}$ are each of them arbitrary.

To prove the theorem in question, it is enough to show (1) that the sections $x=0$, $x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0$; (2) that the sections $z=0$, $x\sqrt{2a}+y\sqrt{2b}+w\sqrt{d}=0$, touch each other.

1. The sections $x=0$, $x\sqrt{2a}+y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0$ of the quadric surface $ax^2+by^2+6xy\sqrt{ab}-cz^2-dw^2=0$ will touch each other if, combining together the equations

$$x=0,\quad y\sqrt{2b}+z\sqrt{c}+w\sqrt{d}=0,\quad by^2-cz^2-dw^2=0,$$

these give a twofold value (pair of equal values) for the ratios $y:z:w$. We in fact have

$$\begin{aligned} by^2-cz^2-dw^2 &= by^2-cz^2-(y\sqrt{2b}+z\sqrt{c})^2,\\ &= -by^2-2cz^2-2yz\sqrt{2bc},\\ &= -(y\sqrt{b}+z\sqrt{2c})^2; \end{aligned}$$

and the right-hand side being a perfect square, the condition of contact is satisfied.

2. In like manner we have the system

$$z=0,\quad x\sqrt{2a}+y\sqrt{2b}+w\sqrt{d}=0,\quad ax^2+by^2+6xy\sqrt{ab}-dw^2=0,$$

which gives

$$\begin{aligned} &ax^2+by^2+6xy\sqrt{ab}-dw^2\\ &\quad= ax^2+by^2+6xy\sqrt{ab}-(x\sqrt{2a}+y\sqrt{2b})^2,\\ &\quad= -ax^2-by^2+2xy\sqrt{ab},\\ &\quad= -(x\sqrt{a}-y\sqrt{b})^2; \end{aligned}$$

and here also, the right-hand side being a perfect square, the condition of contact is satisfied.

Cambridge, November 28, 1863.