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NOTE ON A THEOREM RELATING TO A TRIANGLE, LINE, AND CONIC.

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I FIND, among my papers headed "Generalization of a Theorem of Steiner's," an investigation leading to the following theorem, viz.:

Consider a triangle, a line, and a conic; with each vertex of the triangle join the point of intersection of the line with the polar of the same vertex in regard to the conic; in order that the three joining lines may meet in a point, the line must be a tangent to a curve of the third class; if, however, the conic break up into a pair of lines, or in a certain other case, the curve of the third class will break up into a point, and a conic inscribed in the triangle.

Let the equations of the sides of the triangle be

$$x=0,\quad y=0,\quad z=0,$$

the equation of the conic

$$(a, b, c, f, g, h \between x, y, z)^2=0,$$

and that of the line

$$\lambda x+\mu y+\nu z=0\,;$$

then the polar of the vertex $(y=0,\ z=0)$ has for its equation

$$ax+hy+gz=0\,;$$

it therefore meets the line $\lambda x+\mu y+\nu z=0$ in the point

$$x : y : z = h\nu - g\mu : g\lambda - a\nu : a\mu - h\lambda,$$

and the equation of the line joining this point with the vertex ($y=0$, $z=0$) is $(a\mu-h\lambda)\,y=(g\lambda-a\nu)\,z$. The equations of the three joining lines therefore are

$$\begin{aligned}(a\mu-h\lambda)\,y&=(g\lambda-a\nu)\,z,\\(b\nu-f\mu)\,z&=(h\mu-b\lambda)\,x,\\(c\lambda-g\nu)\,x&=(f\nu-c\mu)\,y,\end{aligned}$$

lines which will meet in a point if

$$(a\mu-h\lambda)(b\nu-f\mu)(c\lambda-g\nu)-(g\lambda-a\nu)(h\mu-b\lambda)(f\nu-c\mu)=0,$$

or, multiplying out and putting as usual

$$\begin{aligned}K&=abc-af^2-bg^2-ch^2+2fgh,\\\mathfrak{A}&=bc-f^2,\ \&c.,\end{aligned}$$

if

$$\left.\begin{aligned}&2\,(abc-fgh)\,\lambda\mu\nu\\&+a\mathfrak{G}\mu\nu^2+a\mathfrak{H}\mu^2\nu\\&+b\mathfrak{H}\nu\lambda^2+b\mathfrak{F}\nu^2\lambda\\&+c\mathfrak{F}\lambda\mu^2+c\mathfrak{G}\lambda^2\mu\end{aligned}\right\}=0,$$

that is, the line must touch a curve of the third class.

If this equation break up into factors, the form must be

$$(\alpha\lambda+\beta\mu+\gamma\nu)(A\mu\nu+B\nu\lambda+C\lambda\mu)=0;$$

that is, we must have

$$\begin{aligned}&A\alpha+B\beta+C\gamma=2\,(abc-fgh),\\&B\alpha=b\mathfrak{H}\,;\quad C\alpha=c\mathfrak{G},\\&C\beta=c\mathfrak{F}\,,\quad A\beta=a\mathfrak{H},\\&A\gamma=a\mathfrak{G},\quad B\gamma=b\mathfrak{F}\,;\end{aligned}$$

and the last six equations give without difficulty

$$\begin{aligned}A&=\frac{ka}{\mathfrak{F}},\qquad \alpha=\frac{1}{k}\mathfrak{G}\mathfrak{H},\\B&=\frac{kb}{\mathfrak{G}},\qquad \beta=\frac{1}{k}\mathfrak{H}\mathfrak{F},\\C&=\frac{kc}{\mathfrak{H}},\qquad \gamma=\frac{1}{k}\mathfrak{F}\mathfrak{G},\end{aligned}$$

where k is arbitrary; the first equation then gives

$$\frac{a\mathfrak{G}\mathfrak{H}}{\mathfrak{F}}+\frac{b\mathfrak{H}\mathfrak{F}}{\mathfrak{G}}+\frac{c\mathfrak{F}\mathfrak{G}}{\mathfrak{H}}=2\,(abc-fgh)\,;$$

or, reducing by the equations $\mathfrak{G}\mathfrak{H} = \mathfrak{A}\mathfrak{F} + aK$, &c., this is

$$\mathfrak{A}a = \mathfrak{B}b + \mathfrak{C}c - 2abc + 2fgh + \left(\frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}}\right) K = 0\,;$$

which, substituting for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ their values, becomes

$$K\left(1 + \frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}}\right) = 0.$$

Hence if $K = 0$, that is, if the conic break up into a pair of lines, or if

$$1 + \frac{a^2}{\mathfrak{F}} + \frac{b^2}{\mathfrak{G}} + \frac{c^2}{\mathfrak{H}} = 0,$$

in either case the equation of the curve of the third class becomes

$$\left(\frac{\lambda}{\mathfrak{F}} + \frac{\mu}{\mathfrak{G}} + \frac{\nu}{\mathfrak{H}}\right)\left(\frac{a}{\mathfrak{F}}\mu\nu + \frac{b}{\mathfrak{G}}\nu\lambda + \frac{c}{\mathfrak{H}}\lambda\mu\right) = 0\,;$$

that is, the curve breaks up into a point, and a conic inscribed in the triangle.

In the case where the conic breaks up into a pair of lines, then we have

$$(a,\ b,\ c,\ f,\ g,\ h \between x,\ y,\ z)^2 = 2\,(px + qy + rz)\,(p'x + q'y + r'z),$$

and thence

$$(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C},\ \mathfrak{F},\ \mathfrak{G},\ \mathfrak{H} \between x,\ y,\ z)^2 = -\{(qr' - q'r)\,x + (rp' - r'p)\,y + (pq' - p'q)\,z\}^2\,;$$

so that the equation in $(\lambda,\ \mu,\ \nu)$ is

$$\{(qr' - q'r)\,\lambda + (rp' - r'p)\,\mu + (pq' - p'q)\,\nu\}$$
$$\{pp'\,(qr' - q'r)\,\mu\nu + qq'\,(rp' - r'p)\,\nu\lambda + rr'\,(pq' - p'q)\,\lambda\mu\} = 0\,;$$

where the point represented by the equation

$$(qr' - q'r)\,\lambda + (rp' - r'p)\,\mu + (pq' - p'q)\,\nu = 0$$

is, of course, the intersection of the two lines.