323.

ON A TACTICAL THEOREM RELATING TO THE TRIADS OF FIFTEEN THINGS.

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THE school-girl problem may be stated as follows:—"With 15 things to form 35 triads, involving all the 105 duads, and such that they can be divided into 7 systems, each of 5 triads containing all the 15 things." A more simple problem is, "With 15 things, to form 35 triads involving all the 105 duads."

In the solution which I formerly gave of the school-girl problem (*Phil. Mag.* vol. XXXVII. 1850, [82]), and which may be presented in the form

	a	*b*	*c*	*d*	*e*	*f*	*g*
abc				35	17	82	64
ade		62	84			15	37
afg		13	57	86	42		
bdf	47		16		38		25
bge	58		23	14		67	
cdg	12	78			56	34	
cef	36	45		27			18

(viz. the things being *a*, *b*, *c*, *d*, *e*, *f*, *g*, 1, 2, 3, 4, 5, 6, 7, 8, the first pentad of triads is *abc*, *d* 35, *e* 17, *f* 82, *g* 64, and so for all the seven pentads of triads), there is obviously a division of the 15 things into (7 + 8) things, viz. the 35 triads are composed 7 of them each of 3 out of the 7 things, and the remaining 28 each of 1 out of the 7 things, and 2 out of the 8 things: or attending only to the 8 things, there are 28 triads each of them containing a duad of the 8 things, but there is no triad consisting of 3 of the 8 things. More briefly, we may say that in the system there is an 8 without 3, that is, there are 8 things such that no triad of them occurs in the system.

I believe, but am not sure, that in all the solutions which have been given of the school-girl problem there is an 8 without 3.

Now, considering the more simple problem, there are of course solutions which have an 8 without 3 (since every solution of the school-girl problem is a solution of the more simple problem): but it is very easy to show that there is no solution which has a 9 without 3. I wish to show that there is in every solution at least a 6 without 3. This being so, there will be (if they all exist) 3 classes of solutions, viz. those which have at most (1) a 6 without 3, (2) a 7 without 3, (3) an 8 without 3. I believe that the first and second classes exist, as well as the third, which is known to do so.

The proposition to be proved is, that given any system of 35 triads involving all the duads of 15 things; there are always 6 things which are a 6 without 3, that is, they are such that no triad of the 6 things is a triad of the system. This will be the case if it is shown that the number of *distinct* hexads which can be formed each of them containing a triad of the system is less than $\left(\frac{15.14.13.12.11.10}{1.2.3.4.5.6} = 5.7.11.13 =\right) 5005$, the entire number of the hexads of 15 things. Now joining to any triad of the system a triad formed out of the remaining 12 things (there are $\frac{12.11.10}{1.2.3} = 4.5.11 = 220$ such triads), we obtain in all $(220 \times 35 =) \, 7700$ hexads, each of them containing a triad of the system. But these 7700 hexads are not all of them distinct. For, first, considering any triad of the system, there are in the system 16 other triads, each of them having no thing in common with the first-mentioned triad. (In fact if e.g. 123 is a triad of the system, then the system, since it contains all the duads, must have besides 6 triads containing 1, 6 triads containing 2, and 6 triads containing 3, and therefore $35-1-6-6-6=16$ triads not containing 1, 2, or 3.) Hence we have $\left(\frac{35.16}{2} =\right) 280$ hexads, each of them composed of two triads of the system; and since each of these hexads can be derived from either of its two component triads, these 280 hexads present themselves twice over among the 7700 hexads.

Secondly, there are in the system seven triads containing each of them the same one thing, e.g.

$$123, \quad 145, \quad 167, \quad 189, \quad 1.10.11, \quad 1.12.13, \quad 1.14.15,$$

containing each of them the thing 1. That is, we have $\left(\frac{7.6}{2} =\right) 21$ pairs such as 123, 145 containing the thing 1, and therefore $(15 \times 21 =) \, 315$ pairs such as $\alpha\beta\gamma$, $\alpha\delta\epsilon$.

And for any such pair, combining with $\alpha\beta\gamma\delta\epsilon$ any one of the remaining 10 things, we have 10 hexads $\alpha\beta\gamma\delta\epsilon\zeta$, each of them derivable from either of the triads $\alpha\beta\gamma$, $\alpha\delta\epsilon$; that is, we have $(315 \times 10 =)$ 3150 hexads which present themselves twice over among the 7700 hexads. The hexads not belonging to one or other of the foregoing classes are derived each of them from a single triad only of the system, and they present themselves once among the 7700 hexads. This number is consequently made up as follows, viz.

280	hexads each	twice =	560
3150	,,	,, =	6300
840	,,	once =	840
4270			7700

or there are in all 4270 distinct hexads; and since this is less than 5005, it follows that there are hexads not containing any triad of the system: there must in fact be $(5005 - 4270 =)$ 735 such hexads. The theorem in question is thus shown to be true.

2, *Stone Buildings, W.C., November* 24, 1862.