312.

ON THE PARTITIONS OF A CLOSE.

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If F, S, E denote the number of faces, summits, and edges of a polyhedron, then, by Euler's well-known theorem,

$$F+S=E+2;$$

and if we imagine the polyhedron projected on the plane of any one face in such manner that the projections of all the summits not belonging to the face fall *within* the face, then we have a partitioned polygon, in which (if P denote the number of component polygons, or say the number of parts) $F=P+1$, or we have

$$P+S=E+1,$$

where S is the number of summits and E the number of edges of the plane figure. I retain for convenience the word *edge*, as having a different initial letter from *summit*.

The formula, however, excludes cases such as that of a polygon divided into two parts by means of an interior polygon wholly detached from it; and in order to extend it to such cases, the formula must be written under the form

$$P+S=E+1+B,$$

where B is the number of breaks of contour, as will be presently explained.

The edges of a polygon are right lines: it might at first sight appear that the theory would not be materially altered by removing this restriction, and allowing the edges to be curved lines; but the fact is that we thus introduce closed figures bounded by two edges, or even by a single edge, or by what I term a mere contour; and we have a new theory, which I call that of the Partitions of a Close.

Several definitions and explanations are required. The words line and curve are used indifferently to denote any path which can be described *currente calamo* without lifting the pen from the paper. A closed curve, not cutting or meeting itself (¹), is called a *contour*. An enclosed space, such that no part of it is shut out from any other part of it, or, what is the same thing, such that any part can be joined with any other part by a line not cutting the boundary, is termed a *close*. The boundary of a close may be considered as the limit of a single contour, or of two or more contours lying wholly within the close. The reason for speaking of a limit will appear by an example. Consider a circle, and within it, but wholly detached from it, a figure of eight; the space interior to the circle but exterior to the figure of eight is a close: its boundary may be considered as the limit of two contours,—the first of them interior to the close, and indefinitely near the circle (in this case we might say the circle itself); the second of them an hour-glass-shaped curve, interior to the close (that is, exterior to the figure of eight) and indefinitely near to the figure of eight. The figure of eight, as being a curve which cuts itself, is not a contour; and in the case in question we could not have said that the boundary of the close consisted of two contours. A similar instance is afforded by a circle having within it two circles exterior to each other, but connected by a line not cutting or meeting itself; or even two points, or, as they may be called, summits, connected by a line not cutting or meeting itself; or, again, a single summit: in each of these cases the boundary of the close may be considered as the limit of two contours. But this explanation once given, we may for shortness speak of the close as bounded by a single contour, or by two or more contours; and I shall throughout do so, instead of using the more precise expression of the boundary being the limit of a contour, or of two or more contours. The excess above unity of the number of the contours which form the boundary of a close is the *break of contour* for such close; in the case of a close bounded by a single contour, the break of contour is zero.

Any point whatever on a curve may be considered as the point of meeting of two curves, or, in the case of a closed curve, as the point where the curve meets itself, but it is not of necessity so considered. A point where a curve cuts or meets itself or any other curve, is a *summit;* each point of termination of an unclosed curve is also a *summit;* any isolated point may be taken to be a *summit*. It follows that, in the case of a closed curve not cutting or meeting itself (that is, a contour), any point or points on the curve may be taken to be summits; but the contour need not have upon it any summit: it is in this case termed a *mere contour*. The curve which is the path from a summit to itself, or to any other summit, is an *edge:* the former case is that of a contour having upon it a single summit, the latter that of an edge having, that is terminated by, two summits, and no more. It is hardly necessary to remark that a contour having upon it two or more summits consists of the same number of edges, and, by what precedes, a contour having upon it a single summit is an edge; but it is to be noted that a contour without any summit upon

¹ It is hardly necessary to add, except in so far as any point whatever of the curve may be considered as a point where the curve meets itself.

it, or mere contour, is *not* an edge. It may be added that an edge does not cut or meet itself or any other edge except at the summit or summits of the edge itself.

Consider now a close bounded by $\beta+1$ mere contours: if for any partitioned close we have P the number of parts, S the number of summits, E the number of edges, B the number of breaks of contour; then, for the unpartitioned close, we have $P=1$, $S=0$, $E=0$, $B=\beta$, and therefore

$$P+S+\beta=E+1+B;$$

and it is to be shown that this equation holds good in whatever manner the close is partitioned. The partitionment is effected by the addition, in any manner, of summits and mere contours, and by drawing edges, any edge from a summit to itself or to another summit. The effect of adding a summit is first to increase S by unity: if the summit added be on a contour, E will be thereby increased by unity; for if the contour is a mere contour, it is not an edge, but becomes so by the addition of the summit; if it is not a mere contour, but has upon it a summit or summits, the addition of the summit will increase by unity the number of edges of the contour. If, on the other hand, the summit added be an isolated one, then the addition of such summit causes a break of contour, or B is increased by unity. Hence the addition of a summit increases by unity S; and it also increases by unity E or else B, that is, it leaves the equation undisturbed. The effect of the addition of a mere contour is to increase P by unity, and also to increase B by unity: it is easy to see that this is the case, whether the new mere contour does or does not contain within it any contour or contours. Hence the addition of a mere contour leaves the equation undisturbed. The effect of drawing an edge is first to increase E by unity; if the edge is drawn from a summit to itself, or from a summit on a contour to another summit on the same contour, then the effect is also to increase P by unity; if, however, the edge is drawn from a summit on a contour to a summit on a different contour, then P remains unaltered, but B is diminished by unity. There are a few special cases, which, although apparently different, are really included in the two preceding ones: thus, if the edge be drawn to connect two isolated summits, these are in fact to be considered as summits belonging to two distinct contours, and the like when a summit on a contour is joined to an isolated summit. And so if there be two or more summits connected together in order, and a new edge is drawn connecting the first and last of them, this is the same as when the edge is drawn through two summits of the same contour. The effect of drawing a new edge is thus to increase E by unity, and also to increase P by unity, or else to diminish B by unity; that is, it leaves the equation undisturbed. Hence the equation $P+S+\beta=E+1+B$, which subsists for the unpartitioned close, continues to subsist in whatever manner the close is partitioned, or it is always true.

In particular, if $\beta=0$, that is, if the original close be bounded by a mere contour, $P+S=E+1+B$; and if, besides, $B=0$, then $P+S=E+1$, which is the ordinary equation in the theory of the partitions of a polygon.

If we consider the surface of a plane as bounded by a mere contour at infinity, then for the infinite plane, $\beta = 0$, or we have $P + S = E + 1 + B$: in the case where the infinite plane is partitioned by a mere contour, $P = 2$, $S = 0$, $E = 0$, $B = 1$ (for the exterior part is bounded by the contour at infinity, and the partitioning contour, that is, for it, $B = 1$), and the equation is thus satisfied. And so for a contour having upon it n summits, $P = 2$, $S = n$, $E = n$, $B = 1$, and the equation is still satisfied: this is the case of the plane partitioned into two parts by means of a single polygon.

The case of a spherical surface is very interesting: the entire surface of the sphere must be considered as a close bounded by 0 contour, or we have $\beta = -1$, and the equation thus becomes $P + S = E + 2 + B$. Thus, if the sphere be divided into two parts by a mere contour, $P = 2$, $S = 0$, $E = 0$, $B = 0$, and the equation is satisfied. And in general, when $B = 0$, then $P + S = E + 2$; or writing F for P, then $F + S = E + 2$, which is Euler's equation for a polyhedron.

2, *Stone Buildings, W.C., March* 8, 1861.