311.

ON A THEOREM OF ABEL'S RELATING TO EQUATIONS OF THE FIFTH ORDER.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 257—263.]

THE following is given (Abel, *Œuvres*, vol. II. p. 253 [Ed. 2, vol. II. p. 266]) as an extract of a letter to M. Crelle:

"Si une équation du cinquième degré, dont les coefficients sont des *nombres rationnels*, est résoluble algébriquement, on peut donner aux racines la forme suivante,

$$x = c + Aa^{\frac{1}{5}}a_1^{\frac{2}{5}}a_2^{\frac{4}{5}}a_3^{\frac{3}{5}} + A_1a_1^{\frac{1}{5}}a_2^{\frac{2}{5}}a_3^{\frac{4}{5}}a^{\frac{2}{5}} + A_2a_2^{\frac{1}{5}}a_3^{\frac{2}{5}}a^{\frac{4}{5}}a_1^{\frac{3}{5}} + A_3a_3^{\frac{1}{5}}a^{\frac{2}{5}}a_1^{\frac{4}{5}}a_2^{\frac{3}{5}},$$

où

$$\begin{aligned}
a &= m + n\sqrt{1+e^2} + \sqrt{h\,(1+e^2+\sqrt{1+e^2})},\\
a_1 &= m - n\sqrt{1+e^2} + \sqrt{h\,(1+e^2-\sqrt{1+e^2})},\\
a_2 &= m + n\sqrt{1+e^2} - \sqrt{h\,(1+e^2+\sqrt{1+e^2})},\\
a_3 &= m - n\sqrt{1+e^2} - \sqrt{h\,(1+e^2-\sqrt{1+e^2})},
\end{aligned}$$

$$\begin{aligned}
A &= K + K'a + K''a_2 + K''aa_2, & A_1 &= K + K'a_1 + K''a_3 + K'''a_1a_3,\\
A_2 &= K + K'a_2 + K''a + K''aa_2, & A_3 &= K + K'a_3 + K''a_1 + K'''a_1a_3.
\end{aligned}$$

Les quantités c, h, e, m, n, K, K', K'', K''' sont des *nombres rationnels*.

"Mais de cette manière l'équation $x^5 + ax + b = 0$ n'est pas résoluble tant que a et b sont des quantités quelconques. J'ai trouvé de pareils théorèmes pour les équations du 7ème, 11ème, 13ème, &c. degré. Fribourg, le 14 Mars, 1826."

The theorem is referred to by M. Kronecker (*Berl. Monatsb.* June 20, 1853), but nowhere else that I am aware of.

www.rcin.org.pl

It is to be noticed that in the expressions for a, a_1, a_2, a_3, the radicals are such that

$$\sqrt{1+e^2}\sqrt{h(1+e^2+\sqrt{1+e^2})}\sqrt{h(1+e^2-\sqrt{1+e^2})}=he(1+e^2),$$

a rational number.

The theorem is given as belonging to numerical equations; but considering it as belonging to literal equations, it will be convenient to change the notation; and in this point of view, and to avoid suffixes and accents, I write

$$x=\theta+A\alpha^{\frac{1}{5}}\beta^{\frac{2}{5}}\gamma^{\frac{4}{5}}\delta^{\frac{3}{5}}+B\beta^{\frac{1}{5}}\gamma^{\frac{2}{5}}\delta^{\frac{4}{5}}\alpha^{\frac{3}{5}}+C\gamma^{\frac{1}{5}}\delta^{\frac{2}{5}}\alpha^{\frac{4}{5}}\beta^{\frac{3}{5}}+D\delta^{\frac{1}{5}}\alpha^{\frac{2}{5}}\beta^{\frac{4}{5}}\gamma^{\frac{3}{5}},$$

where

$$\alpha=m+n\sqrt{\Theta}+\sqrt{p+q\sqrt{\Theta}},$$

$$\beta=m-n\sqrt{\Theta}+\sqrt{p-q\sqrt{\Theta}},$$

$$\gamma=m+n\sqrt{\Theta}-\sqrt{p+q\sqrt{\Theta}},$$

$$\delta=m-n\sqrt{\Theta}-\sqrt{p-q\sqrt{\Theta}};$$

the radicals being connected by

$$\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}}=s,$$

and where

$$A=K+L\alpha+M\gamma+N\alpha\gamma,\quad B=K+L\beta+M\delta+N\beta\delta,$$

$$C=K+L\gamma+M\alpha+N\alpha\gamma,\quad D=K+L\delta+M\beta+N\beta\delta,$$

in which equations θ, m, n, p, q, Θ, s, K, L, M, N are rational functions of the elements of the given quintic equation.

The basis of the theorem is, that the expression for x has only the five values which it acquires by giving to the quintic radicals contained in it their five several values, and does not acquire any new value by substituting for the quadratic radicals their several values. For, this being so, x will be the root of a rational quintic; and conversely.

Now attending to the equation

$$\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}}=s,$$

the different admissible values of the radicals are

$$\begin{array}{lll} \sqrt{\Theta}, & \sqrt{p+q\sqrt{\Theta}}, & \sqrt{p-q\sqrt{\Theta}}, \\ -\sqrt{\Theta}, & \sqrt{p-q\sqrt{\Theta}}, & -\sqrt{p+q\sqrt{\Theta}}, \\ \sqrt{\Theta}, & -\sqrt{p+q\sqrt{\Theta}}, & -\sqrt{p-q\sqrt{\Theta}}, \\ -\sqrt{\Theta}, & -\sqrt{p-q\sqrt{\Theta}}, & \sqrt{p+q\sqrt{\Theta}}, \end{array}$$

www.rcin.org.pl

corresponding to the systems

$$\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \beta, & \gamma, & \delta, & \alpha, \\ \gamma, & \delta, & \alpha, & \beta, \\ \alpha, & \beta, & \gamma, & \delta, \end{array}$$

of the roots $\alpha, \beta, \gamma, \delta$; i.e. the effect of the alteration of the values of the quadratic radicals is merely to cyclically permute the roots $\alpha, \beta, \gamma, \delta$; and observing that any such cyclical permutation gives rise to a like cyclical permutation of A, B, C, D, the alteration of the quadratic radicals produces no alteration in the expression for x.

The quantities $\alpha, \beta, \gamma, \delta$ are the roots of a rational quartic. If, solving the quartic by Euler's method, we write

$$\alpha = m + \sqrt{F} + \sqrt{G} + \sqrt{H}, \qquad \sqrt{FGH} = \nu, \text{ a rational function,}$$

$$\beta = m - \sqrt{F} + \sqrt{G} - \sqrt{H},$$

$$\gamma = m + \sqrt{F} - \sqrt{G} - \sqrt{H},$$

$$\delta = m - \sqrt{F} - \sqrt{G} + \sqrt{H},$$

then the expressions for F, G, H in terms of the roots are

$$(\alpha + \gamma - \beta - \delta)^2, \quad (\alpha + \beta - \gamma - \delta)^2, \quad (\alpha + \delta - \beta - \gamma)^2,$$

which are the roots of a cubic equation

$$u^3 - \lambda u^2 + \mu u - \nu^2 = 0,$$

where λ, μ, ν are given rational functions of the coefficients of the quartic. We have

$$\sqrt{G} + \sqrt{H} = \sqrt{(\sqrt{G} + \sqrt{H})^2} = \sqrt{G + H + 2\sqrt{GH}} = \sqrt{\lambda - F + \frac{2\nu}{F}\sqrt{F}};$$

so that, taking $\Theta = F$, the last-mentioned expressions for $\alpha, \beta, \gamma, \delta$ will be of the assumed form

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \quad \&c.$$

The equation

$$\sqrt{\Theta}\sqrt{p + q\sqrt{\Theta}}\sqrt{p - q\sqrt{\Theta}} = s$$

thus becomes

$$\sqrt{F}\sqrt{(G - H)^2} = s, \text{ or } F(G - H)^2 = s^2;$$

that is,

$$-F^3 + F(F^2 + G^2 + H^2) - 2FGH = s^2;$$

or, what is the same thing, and putting Θ for F,

$$-\lambda\Theta^2 + (\lambda^2 - \mu)\Theta - 3\nu = s^2.$$

www.rcin.org.pl

Hence in order that the roots of the quartic may be of the assumed form,

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \text{ \&c.},$$

where m, p, q, Θ are rational, and where also

$$\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}} = s, \text{ a rational function,}$$

the necessary and sufficient conditions are that the quartic should be such that the reducing cubic

$$u^3 - \lambda u^2 + \mu u - \nu^2 = 0$$

(whose roots are $(\alpha+\beta-\gamma-\delta)^2$, $(\alpha+\gamma-\beta-\delta)^2$, $(\alpha+\delta-\beta-\gamma)^2$) may have *one rational root* Θ, and moreover that the function

$$-\lambda\Theta^2 + (\lambda^2 - \mu)\Theta - 3\nu$$

shall be the *square of a rational function s*. This being so, the roots of the quartic will be of the assumed form

$$\alpha = m + \sqrt{\Theta} + \sqrt{p + q\sqrt{\Theta}}, \text{ \&c.};$$

and from what precedes, it is clear that any function of the roots of the quartic which remains unaltered by the cyclical substitution $\alpha\beta\gamma\delta$, or what is the same thing, any function of the form

$$\phi(\alpha, \beta, \gamma, \delta) + \phi(\beta, \gamma, \delta, \alpha) + \phi(\gamma, \delta, \alpha, \beta) + \phi(\delta, \alpha, \beta, \gamma)$$

will be a rational function of m, Θ, p, q, s, and consequently of the coefficients of the quartic. The above are the conditions in order that a quartic equation may be of the Abelian form.

It may be as well to remark that, assuming only the system of equations

$$\alpha = m + \sqrt{\Theta} + \sqrt{\Upsilon},$$

$$\beta = m - \sqrt{\Theta} + \sqrt{\Upsilon'},$$

$$\gamma = m + \sqrt{\Theta} - \sqrt{\Upsilon},$$

$$\delta = m - \sqrt{\Theta} - \sqrt{\Upsilon'},$$

then any rational function of α, β, γ, δ which remains unaltered by the cyclical substitution $\alpha\beta\gamma\delta$ will be a rational function of Θ, $\Upsilon + \Upsilon'$, $\Upsilon\Upsilon'$, $\sqrt{\Upsilon\Upsilon'}(\Upsilon - \Upsilon')$, $\sqrt{\Theta}(\Upsilon - \Upsilon')$, $\sqrt{\Theta}\sqrt{\Upsilon\Upsilon'}$. In fact, suppose such a function contains the term

$$(\sqrt{\Theta})^\alpha (\sqrt{\Upsilon})^\beta (\sqrt{\Upsilon'})^\gamma;$$

then it will contain the four terms

$$\begin{aligned}
&(\sqrt{\Theta})^\alpha (\sqrt{\Upsilon})^\beta (\sqrt{\Upsilon'})^\gamma,\\
&(-\sqrt{\Theta})^\alpha (\sqrt{\Upsilon'})^\beta (-\sqrt{\Upsilon})^\gamma,\\
&(\sqrt{\Theta})^\alpha (-\sqrt{\Upsilon})^\beta (-\sqrt{\Upsilon'})^\gamma,\\
&(-\sqrt{\Theta})^\alpha (-\sqrt{\Upsilon'})^\beta (\sqrt{\Upsilon})^\gamma,
\end{aligned}$$

www.rcin.org.pl

which together are

$$(\sqrt{\Theta})^\alpha \{(1+(-)^{\beta+\gamma}1)(\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma + (-)^\alpha[(-)^\beta 1 + (-)^\gamma 1](\sqrt{\Upsilon})^\gamma(\sqrt{\Upsilon'})^\beta\},$$

an expression which vanishes unless $(-)^\beta$, $(-)^\gamma$ are both positive or both negative. The forms to be considered are therefore

$(-)^\alpha$,	$(-)^\beta$,	$(-)^\gamma$
$+$	$+$	$+$
$-$	$+$	$+$
$+$	$-$	$-$
$-$	$-$	$-$

The first form is

$$(\sqrt{\Theta})^\alpha \{(\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma + (\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma\},$$

which, α, β, γ being each of them even, is a rational function of Θ, $\Upsilon+\Upsilon'$, $\Upsilon\Upsilon'$.

The second form is

$$(\sqrt{\Theta})^\alpha \{(\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma - (\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma\},$$

which, α being odd and β and γ each of them even, is the product of such a function into $\sqrt{\Theta}(\Upsilon-\Upsilon')$.

The third form is

$$(\sqrt{\Theta})^\alpha \{(\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma - (\sqrt{\Upsilon})^\gamma(\sqrt{\Upsilon'})^\beta\},$$

which, α being even and β and γ each of them odd, is the product of such a function into $\sqrt{\Upsilon\Upsilon'}(\Upsilon-\Upsilon')$.

And the fourth form is

$$(\sqrt{\Theta})^\alpha \{(\sqrt{\Upsilon})^\beta(\sqrt{\Upsilon'})^\gamma + (\sqrt{\Upsilon})^\gamma(\sqrt{\Upsilon'})^\beta\},$$

which, α, β, γ being each of them odd, is the product of such a function into $\sqrt{\Theta}(\Upsilon-\Upsilon')$.

Hence if $\Upsilon = p + q\sqrt{\Theta}$, $\Upsilon' = p - q\sqrt{\Theta}$, and $\sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}} = s$, then

$$\Theta,\quad \Upsilon+\Upsilon'(=2p),\quad \Upsilon\Upsilon'(=p^2-q^2\Theta),\quad \sqrt{\Upsilon\Upsilon'}(\Upsilon-\Upsilon')\left(=\frac{2q}{s}\right),$$
$$\sqrt{\Theta}(\Upsilon-\Upsilon')(=2q\Theta),\text{ and } \sqrt{\Theta}\sqrt{\Upsilon\Upsilon'}(=s)$$

are respectively rational functions. This is the *à posteriori* verification, that with the system of equations

$$\alpha = m + \sqrt{\Theta} + \sqrt{p+q\sqrt{\Theta}},\ \&\text{c.},\quad \sqrt{\Theta}\sqrt{p+q\sqrt{\Theta}}\sqrt{p-q\sqrt{\Theta}} = s,$$

any function

$$\phi(\alpha,\beta,\gamma,\delta)+\phi(\beta,\gamma,\delta,\alpha)+\phi(\gamma,\delta,\alpha,\beta)+\phi(\delta,\alpha,\beta,\gamma)$$

is a rational function.

www.rcin.org.pl

The coefficients of the quintic equation for x must of course be of the form just mentioned; that is, they must be functions of α, β, γ, δ, which remain unaltered by the cyclic substitution $\alpha\beta\gamma\delta$. To form the quintic equation, I write

$$\theta - x = a,$$

$$A\alpha^{\frac{1}{5}}\beta^{\frac{2}{5}}\gamma^{\frac{4}{5}}\delta^{\frac{3}{5}} = b,\quad D\delta^{\frac{1}{5}}\alpha^{\frac{2}{5}}\beta^{\frac{4}{5}}\gamma^{\frac{3}{5}} = c,\quad B\beta^{\frac{1}{5}}\gamma^{\frac{2}{5}}\delta^{\frac{4}{5}}\alpha^{\frac{3}{5}} = d,\quad C\gamma^{\frac{1}{5}}\delta^{\frac{2}{5}}\alpha^{\frac{4}{5}}\beta^{\frac{3}{5}} = e;$$

then we have

$$0 = a + b + c + d + e,$$

and the quintic equation is

$$f1\, f\omega\, f\omega^2\, f\omega^3\, f\omega^4 = 0,$$

where ω is an imaginary fifth root of unity, and

$$f\omega = a + b\omega + c\omega^2 + d\omega^3 + e\omega^4.$$

We have

$$f\omega\, f\omega^4 = \Sigma a^2 + (\omega + \omega^4)\, \Sigma' ab + (\omega^2 + \omega^3)\, \Sigma' ac,$$

$$f\omega^2 f\omega^3 = \Sigma a^2 + (\omega^2 + \omega^3)\, \Sigma' ab + (\omega^2 + \omega^3)\, \Sigma' ac,$$

where Σ' is Mr Harley's cyclical symbol, viz.

$$\Sigma' ab = ab + bc + cd + de + ea;$$

and so in other cases, the order of the cycle being always *abcde*. This gives

$$f\omega\, f\omega^2\, f\omega^3\, f\omega^4 = \Sigma a^4 + \Sigma a^2b^2 - \Sigma a^3b + 2\Sigma a^2bc - \Sigma abcd - 5\Sigma' a^2(be + cd);$$

and multiplying by $f1$, $= \Sigma a$, and equating to zero, the result is found to be

$$\Sigma a^5 - 5abcde - 5\Sigma' a^3(be + cd) + 5\Sigma' a\,(b^2e^2 + c^2d^2) = 0;$$

or arranging in powers of a, this is

$$\left.\begin{array}{l} a^5 \\ + a^3 . \quad - 5\,(be \quad + cd) \\ + a^2 . \quad\ \ 5\,(b\,c^2 \quad + c\,e^2 \quad + e\,d^2 + d\,b^2) \\ + a\ . \left\{\begin{array}{l} \ \ 5\,(b^3c \quad + c^3e \quad + e^3d + d^3\,b) \\ + 5\,(b^2e^2 \ + c^2d^2 \ - becd) \end{array}\right. \\ + \left\{\begin{array}{l} \qquad b^5 + c^5 + e^5 + d^5 \\ - 5\,(b^3de + c^3bd + e^3cb + d^3ec) \\ + 5\,(bd^2e^2 + cb^2d^2 + ec^2b^2 + de^2c^2) \end{array}\right. \end{array}\right\} = 0,$$

the several coefficients being, it will be observed, cyclical functions to the cycle b, c, e, d.

www.rcin.org.pl

Putting for a its value $-(x-\theta)$, and for b, c, d, e their values, the quintic equation in x is

$$\left.\begin{array}{l}
(x-\theta)^5 \\
+(x-\theta)^3 . \quad -5\,(AC \quad + BD)\,\alpha\beta\gamma\delta \\
+(x-\theta)^2 . \quad -5\,(A^2B\gamma\delta \quad + B^2C\delta\alpha \quad + C^2D\alpha\beta \quad + D^2A\beta\gamma\ )\,\alpha\beta\gamma\delta \\
+(x-\theta) \left\{\begin{array}{l} -5\,(A^3D\beta\gamma^2\delta + B^3A\gamma\delta^2\alpha + C^3B\delta\alpha^2\beta + D^3C\alpha\beta^2\gamma)\,\alpha\beta\gamma\delta \\ +5\,(A^2C^2 \quad + B^2D^2 \quad - ABCD)\,\alpha^2\beta^2\gamma^2\delta^2 \end{array}\right. \\
+ \left\{\begin{array}{l} \quad (A^5\beta\gamma^3\delta^2 \quad + B^5\gamma\delta^3\alpha^2 \quad + C^5\delta\alpha^3\beta^2 \quad + D^5\alpha\beta^3\gamma^2\ )\,\alpha\beta\gamma\delta \\ -5\,(A^3BC\gamma\delta \quad + B^3CD\delta\alpha \quad + C^3DA\alpha\beta \quad + D^3AB\beta\gamma)\,\alpha^2\beta^2\gamma^2\delta^2 \\ +5\,(AB^2C^2\alpha\delta + BC^2D^2\beta\alpha + CD^2A^2\gamma\beta + DE^2A^2\delta\alpha)\,\alpha^2\beta^2\gamma^2\delta^2 \end{array}\right.
\end{array}\right\} = 0,$$

where as before

$$\begin{aligned}
A &= K + L\alpha + M\gamma + N\alpha\gamma, \\
B &= K + L\beta + M\delta + N\beta\delta, \\
C &= K + L\gamma + M\alpha + N\gamma\alpha, \\
D &= K + L\delta + M\beta + N\delta\beta;
\end{aligned}$$

and the coefficients of the quintic equation are, as they should be, cyclical functions with the cycle $\alpha\beta\gamma\delta$.

2, *Stone Buildings, W.C., February* 10, 1861.

www.rcin.org.pl