309.

NOTE ON THE THEORY OF DETERMINANTS.

[From the *Philosophical Magazine*, vol. XXI. (1861), pp. 180—185.]

THE following mode of arrangement of the developed expression of a determinant had presented itself to me as a convenient one for the calculation of a rather complicated determinant of the fifth order; but I have since found that it is in effect given, although in a much less compendious form, in a paper by J. N. Stockwell, "On the Resolution of a System of Symmetrical Equations with Indeterminate Coefficients," Gould's *Ast. Journal*, No. 139 (Cambridge, U. S., Sept. 10, 1860).

Suppose that the determinant

$$\left|\begin{matrix} 11, & 12, & 13 \\ 21, & 22, & 23 \\ 31, & 32, & 33 \end{matrix}\right|$$

is represented by $\{123\}$, and so for a determinant of any order $\{123 \ldots n\}$.

Let $|\,1\,|$, $|\,2\,|$, $|\,12\,|$, $|\,123\,|$, &c., denote as follows: viz.

$$|\,1\,| = 11, \quad |\,2\,| = 22, \text{ \&c.}$$

$$|\,12\,| = 12 \,.\, 21,$$

$$|\,123\,| = 12 \,.\, 23 \,.\, 31,$$

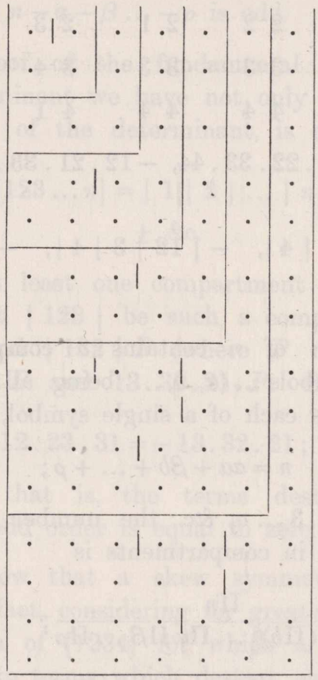
$$\text{\&c.},$$

where it is to be noticed that, with the same two symbols, e.g. 1 and 2, there is but one distinct expression $|\,12\,|$ (in fact $|\,21\,| = 21 \,.\, 12 = |\,12\,|$); with the same three symbols, 1, 2, 3, there are two distinct expressions, $|\,123\,|$ $(= 12 \,.\, 23 \,.\, 31)$ and $|\,132\,|$ $(= 13 \,.\, 32 \,.\, 21)$; and generally with the same m symbols $1, 2, 3 \ldots m$, there are $1 \,.\, 2 \,.\, 3 \ldots (m-1)$ distinct expressions $\vert 123 \ldots m \vert$, which are obtained by permuting in every possible manner all but one of the m symbols.

This being so, and writing for greater simplicity $\vert 1 \vert 2 \vert$ to denote the product $\vert 1 \vert \times \vert 2 \vert$, and so in general, the values of the determinants $\{12\}$, $\{123\}$, $\{1234\}$, $\{12345\}$, &c. are as follows: viz.

		No. of terms +	No. of terms −
$\{12\} =$	$+\vert 1 \vert 2 \vert$	1	
	$-\vert 1 \ \ 2 \vert$		1
		1 +	1 = 2
$\{123\} =$	$+\vert 1 \vert 2 \vert 3 \vert$ . . .	1	
	$-\vert 1 \ \ 2 \vert 3 \vert$ . . .		3
	$+\vert 1 \ \ 2 \ \ 3 \vert$ . . .	2	
		3 +	3 = 6
$\{1234\} =$	$+\vert 1 \vert 2 \vert 3 \vert 4 \vert$ . .	1	
	$-\vert 1 \ \ 2 \vert 3 \vert 4 \vert$ . .		6
	$+\vert 1 \ \ 2 \ \ 3 \vert 4 \vert$ . .	8	
	$+\vert 1 \ \ 2 \vert 3 \ \ 4 \vert$ . .	3	
	$-\vert 1 \ \ 2 \ \ 3 \ \ 4 \vert$ . .		6
		12 +	12 = 24
$\{12345\} =$	$+\vert 1 \vert 2 \vert 3 \vert 4 \vert 5 \vert$.	1	
	$-\vert 1 \ \ 2 \vert 3 \vert 4 \vert 5 \vert$.		10
	$+\vert 1 \ \ 2 \ \ 3 \vert 4 \vert 5 \vert$.	20	
	$+\vert 1 \ \ 2 \vert 3 \ \ 4 \vert 5 \vert$	15	
	$-\vert 1 \ \ 2 \ \ 3 \ \ 4 \vert 5 \vert$.		30
	$-\vert 1 \ \ 2 \ \ 3 \vert 4 \ \ 5 \vert$.		20
	$+\vert 1 \ \ 2 \ \ 3 \ \ 4 \ \ 5 \vert$.	24	
		60 +	60 = 120

where, as regards the signs, it is to be observed that there is a sign − for each compartment $\vert \quad \vert$ containing an even number of symbols; thus in the expression for $\{1234\}$, the terms $\vert 1\ 2 \vert 3\ 4 \vert$ have the sign $- - = +$, and the terms $\vert 1\ 2\ 3\ 4 \vert$ the sign −. Or, what comes to the same thing; when n is even, the sign is + or − according as the number of compartments is even or odd; and contrariwise when n is odd. As regards the remaining part of the expression, this merely exhibits the partitions of a set of n things; and the formulæ for the several determinants up to the determinant of a given order are all of them obtained by means of the form



which is carried up to the order 7, but which can be further extended without any difficulty whatever.

It is perhaps hardly necessary, but I give at full length the expressions of the determinant of the third order: this is

$$\begin{array}{rl} \{123\} = & \;\;\, | 1 | 2 | 3 | \\ & \left.\begin{array}{l} - | 1 \;\; 2 | 3 | \\ - | 2 \;\; 3 | 1 | \\ - | 3 \;\; 1 | 2 | \end{array}\right\} \\ & \left.\begin{array}{l} + | 1 \;\; 2 \;\; 3 | \\ + | 1 \;\; 3 \;\; 2 | \end{array}\right\}; \end{array}$$

and by writing down in like manner the expression for the twenty-four terms of the determinant of the fourth order, the notation will become perfectly clear.

The formula hardly requires a demonstration. The terms of a determinant $\{123 \ldots n\}$, for example the determinant $\{1234\}$, are obtained by permuting in every possible manner the symbols in either column, say the second column, of the arrangement

$$\begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{array}$$

and prefixing the sign (+ or −) of the arrangement; and the resulting arrangements, for instance

$$\begin{matrix} +1\,1, & -1\,2, & -1\,2, \\ 2\,2 & 2\,1 & 2\,3 \\ 3\,3 & 3\,3 & 3\,4 \\ 4\,4 & 4\,4 & 4\,1 \end{matrix}$$

are interpreted either into $+11\,.\,22\,.\,33\,.\,44$, $-12\,.\,21\,.\,33\,.\,44$, $-12\,.\,23\,.\,34\,.\,41$, or in the notation of the formula, into

$$+\,|\,1\,|\,2\,|\,3\,|\,4\,|,\quad -\,|\,12\,|\,3\,|\,4\,|,\quad -\,|\,1\,2\,3\,4\,|;$$

and so in general.

Suppose that any partition of n contains α compartments each of a symbols, β compartments each of b symbols ... (a, b, ... being all of them different and greater than unity), and ρ compartments each of a single symbol, we have

$$n = \alpha a + \beta b + \ldots + \rho\,;$$

and writing, as usual, $\Pi a = 1\,.\,2\,.\,3 \ldots a$, &c., the number of ways in which the symbols 1, 2, 3, ... n, can be so arranged in compartments is

$$\frac{\Pi n}{(\Pi a)^{\alpha}\,(\Pi b)^{\beta} \ldots \Pi\alpha\,\Pi\beta \ldots \Pi\rho}\,;$$

but each such arrangement gives $(\Pi\,(a-1))^{\alpha}\,.\,(\Pi\,(b-1))^{\beta}$ terms of the determinant, and the corresponding number of terms therefore is

$$\frac{\Pi n}{a^{\alpha} b^{\beta} \ldots \Pi\alpha\,\Pi\beta \ldots \Pi\rho}\,.$$

The whole number of terms of the determinant is Πn, and we have thus the theorem

$$1 = \Sigma\,\frac{1}{a^{\alpha} b^{\beta} \ldots \Pi\alpha\,\Pi\beta \ldots \Pi\rho}\,,$$

in which the summation corresponds to all the different partitions $n = \alpha a + \beta b, \ldots + \rho$, where a, b, ... are all of them different and greater than unity; a theorem given in Cauchy's *Mémoire sur les Arrangements* &c., 1844. But it is to be noticed also that, the number of the positive and negative terms being equal, we have besides

$$0 = \Sigma\,\frac{(-)^{\alpha(a-1)+\beta(b-1)+\ldots}}{a^{\alpha} b^{\beta} \ldots \Pi\alpha\,\Pi\beta \ldots \Pi\rho}\,;$$

or, what is the same thing,

$$0 = \Sigma\,\frac{(-)^{n-\alpha-\beta\ldots-\rho}}{a^{\alpha} b^{\beta} \ldots \Pi\alpha\,\Pi\beta \,.\,.\, \Pi\rho}\,;$$

and thence also

$$\tfrac{1}{2} = \Sigma\,\frac{1}{a^{\alpha} b^{\beta} \ldots \Pi\alpha\,\Pi\beta \ldots \Pi\rho}\,;$$

where, as before, $n = \alpha a + \beta b \ldots + \rho$ (a, b, ... being all different and greater than unity); but the summation is restricted either to the partitions for which $n - \alpha - \beta \ldots - \rho$ is even, or else to those for which $n - \alpha - \beta \ldots - \rho$ is odd.

The formula affords a proof of the fundamental property of skew symmetrical determinants. In such a determinant we have not only $12 = -21$, &c., but also $11 = 0$, &c. Suppose that n, the order of the determinant, is odd; then in each line of the expression

$$\{123 \ldots n\} = |\ 1\ |\ 2\ |\ \ldots\ |\ n\ |$$
$$\pm \text{ \&c.}$$

of the determinant, there is at least one compartment $|\ 1\ |$ or $|\ 123\ |$ &c. containing an odd number of symbols: let $|\ 123\ |$ be such a compartment, then the determinant contains the terms $|\ 123\ | P$ and $|\ 132\ | P$ (where P represents the remaining compartments), that is, $12 . 23 . 31 . P$ and $13 . 32 . 21 . P$. But in virtue of the relations $12 = -21$, &c., we have

$$12 . 23 . 31 = -13 . 32 . 21;$$

and so in all similar cases; that is, the terms destroy each other, or the skew symmetrical determinant of an odd order is equal to zero.

The like considerations show that a skew symmetrical determinant of an even order is a perfect square. In fact, considering for greater simplicity the case $n = 4$, any line in the foregoing expression of $\{1234\}$ for which a compartment contains an odd number of symbols, gives rise to terms which destroy each other, and may be omitted. The expression thus reduces itself to

$$\{1234\} = +\ |\ 12\ |\ 34\ | \quad 3 \text{ terms}$$
$$-\ |\ 12 \quad 34\ | \quad 6 \text{ terms},$$

which is in fact the square of

$$12 . 34 + 13 . 42 + 14 . 23;$$

for the square of a term, say $12 . 34$, is $\overline{12}^2 . \overline{34}^2$ or $12 . 21 . 34 . 43$, that is, $|\ 12\ |\ 34\ |$, and the double of the product of two terms, say $12 . 34$ and $13 . 42$, is $2 . 12 . 34 . 13 . 42$, or $-12 . 24 . 43 . 31 - 13 . 34 . 42 . 21$, that is $-\ |\ 1243\ | - |\ 1342\ |$, and so for the other similar terms, and we have

$$\{1234\} = (12 . 34 + 13 . 42 + 14 . 23)^2:$$

and so in general, n being any even number, the skew symmetrical determinant $\{123 \ldots n\}$ is equal to the square of the Pfaffian $12 \ldots n$, where the law of these Pfaffian functions is

$$1234 \quad = 12 . 34 \quad + 13 . 42 \quad + 14 . 23$$
$$123456 = 12 . 3456 + 13 . 4562 + 14 . 5623 + 15 . 6234 + 16 . 2345,$$

where, in the second equation, 3456, &c. are Pfaffians, viz. $3456 = 34 . 56 + 35 . 64 + 36 . 45$; and so on.

2, *Stone Buildings, W.C., December* 28, 1860.