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BRIEF NOTES

On a certain generalization of the concept of simple material

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THE aim of the note is to give an outline of a general approach to the concept of ideal materials, for which the principle of local action holds. To meet this aim the constitutive equations are postulated in the form of restrictions imposed on the histories of the stress and the local deformation. The classes of materials defined by the resulting constitutive equations are referred to as the generalized simple materials. The latter include, as the special cases, the known classes of simple materials, rate-type materials, elastic-plastic materials or materials with internal variables, as well as those classes of materials which have not been investigated up to now.

1. Generalized simple materials

LET $\boldsymbol{\chi}(\mathbf{X}, t)$ and $\mathbf{T}(\mathbf{X}, t)$, $\mathbf{X} \in \boldsymbol{\varkappa}(\mathscr{B})$, $t \in R$, be the deformation function and the Cauchy stress tensor, respectively, in the body $\mathscr{B}$(1). We are to define the ideal material at an arbitrary but fixed particle $X \in \mathscr{B}$ as the restriction imposed on the stresses at the particle X and on the motion of the body $\mathscr{B}$. We shall also connect this restriction with the present time instant $t \in R$ through the relation

$$(1.1) \qquad \mathop{\mathscr{G}_{\varkappa}}_{\substack{\mathbf{Z} \in \varkappa(\mathscr{B}) \\ \sigma, \tau \in (-\infty, t]}} \big(X, t; \boldsymbol{\chi}(\mathbf{Z}, \sigma), \mathbf{T}(\mathbf{X}, \tau)\big) = \mathbf{0},$$

where $\mathscr{G}_{\varkappa} = (\mathscr{G}^1_{\varkappa}, \ldots, \mathscr{G}^n_{\varkappa})$ is a known functional. We assume that the form of the functional relation (1.1) is restricted by the principle of the local action and that of the material frame indifference. The former principle gives

$$\mathop{\mathscr{G}_{\varkappa}}_{\substack{\mathbf{Z} \in \varkappa(\;) \\ \sigma, \tau (-\infty, t]}} \big(X, t; \boldsymbol{\chi}(\mathbf{X}, \sigma) + \nabla\boldsymbol{\chi}(\mathbf{X}, \sigma)(\mathbf{Z} - \mathbf{X}), \mathbf{T}(\mathbf{X}, \tau)\big) = \mathbf{0},$$

and after that the latter leads to the relation(2)

$$(1.2) \qquad \mathop{\mathfrak{h}_{\mathbf{K}}}_{\sigma, \tau \in (-\infty, t]} \big(X; \mathbf{F}(\sigma), \mathbf{T}(\tau)\big) = 0,$$

where $\mathfrak{h}_{\mathbf{K}} = (\mathfrak{h}^1_{\mathbf{K}}, \ldots, \mathfrak{h}^n_{\mathbf{K}})$ is a known functional and $\mathbf{F}(\sigma)$ is a local deformation from the

(1) The denotations are based on those given in [1].

(2) In what follows we use the denotations $\mathbf{F}(\sigma) \equiv \mathbf{F}(\mathbf{X}, \sigma)$, $\mathbf{T}(\tau) = \mathbf{T}(\mathbf{X}, \tau)$.

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fixed local configuration **K**, $\det \mathbf{F}(\sigma) > 0$. The principle of the material frame indifference also implies the condition

$$\forall(\mathbf{Q}(s))[\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X; \mathbf{Q}(\sigma)\mathbf{F}(\sigma), \mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^T(\tau)) = \mathbf{0}], \tag{1.3}$$

which holds for any pair $(\mathbf{F}(\sigma), \mathbf{T}(\tau))$ satisfying Eq. (1.2), and where $\mathbf{Q}(s)$, $s \in (-\infty, t]$, is an orthogonal 3×3 matrix function. We shall also assume that the components $\mathfrak{h}^1_{\mathbf{K}}, \dots, \mathfrak{h}^n_{\mathbf{K}}$ of the vector function $\mathfrak{h}_{\mathbf{K}}$ transform as components of scalars, tensors or tensor densities in the physical space.

Provided that the functional $\mathfrak{h}_{\mathbf{K}}$ is known and the condition (1.3) holds, Eq. (1.2) represents a definition of a certain class (possible empty) of what will be called the generalized simple materials. At the same time the functional $\mathfrak{h}_{\mathbf{K}}$ is said to be the constitutive functional for this class of materials. For non pure mechanical theories the functional $\mathfrak{h}_{\mathbf{K}}$ can also depend on other variables than $\mathbf{F}(\sigma)$ and $\mathbf{T}(\tau)$ (cf. Sect. 3).

2. Some general formulas

Now, we shall show that the known concepts and definitions concerning simple materials can be easily reformulated to a form which is also suitable for the generalized simple materials.

Substituting the polar decomposition $\mathbf{F}(\sigma) = \mathbf{R}(\sigma)\mathbf{U}(\sigma)$ into Eq. (1.2), assuming that the rotation $\mathbf{R}(\sigma)$ is not constrained when $\sigma \in (-\infty, t]$, and denoting $\mathbf{Q}(\sigma) \equiv \mathbf{R}^T(\sigma)$, we obtain the reduced form of the constitutive equation of a generalized simple material

$$\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X; \mathbf{U}(\sigma), \mathbf{R}^T(\tau)\mathbf{T}(\tau)\mathbf{R}(\tau)) = 0. \tag{2.1}$$

Eq. (2.1) has to hold for any history of rotation up to the time instant t. Alternative forms of the reduced constitutive equations we obtain substituting into (1.3): $\mathbf{Q}(\tau) = \mathbf{R}^T(t)\mathbf{R}^T_{(t)}(\tau)$ and $\mathbf{Q}(\sigma)\mathbf{F}(\sigma) = \mathbf{R}^T(t)\mathbf{R}^T_{(t)}(\sigma)\,\mathbf{F}_{(t)}(\sigma)\mathbf{F}(t) = \mathbf{R}^T(t)\mathbf{U}_{(t)}(\sigma)\mathbf{R}(t)\mathbf{U}(t)$, cf. [1]. Thus, we arrive at

$$\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X; \mathbf{U}^*_{(t)}(\sigma)\mathbf{U}(t), \mathbf{R}^T(t)\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)\mathbf{R}(t)) = \mathbf{0}, \tag{2.2}$$

where $\mathbf{U}^*_{(t)}(\sigma) = \mathbf{R}^T(t)\mathbf{U}_{(t)}(\sigma)\mathbf{R}(t)$. By virtue of Eq. (1.3) we also obtain

$$\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X; \mathbf{U}_{(t)}(\sigma)\mathbf{R}(t)\mathbf{U}(t), \mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)) = \mathbf{0}. \tag{2.3}$$

Eqs. (2.2) and (2.3) remain valid if we replace the present time instant t in $\mathbf{U}_{(t)}$, $\mathbf{R}_{(t)}$ by an arbitrary time instant s, $s \in (-\infty, t]$.

Let $\varrho_{\mathbf{K}}(X)$ denote the mass density at the particle X in the local configuration **K**. Two particles $\hat{X}$, $\check{X}$ will be called isomorphic if the local configurations $\hat{\mathbf{K}}$, $\check{\mathbf{K}}$ exist such that

$$\varrho_{\hat{\mathbf{K}}}(\hat{X}) = \varrho_{\check{\mathbf{K}}}(\check{X}), \quad \underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\hat{\mathbf{K}}}}(\hat{X}; \cdot, \cdot) = \underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\check{\mathbf{K}}}}(\check{X}; \cdot, \cdot). \tag{2.4}$$

This definition also enables us to define the concept of the homogeneous body made of the generalized simple material.

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By the isotropy group $g_{\mathbf{K}}$ of the constitutive functional $\mathfrak{h}_{\mathbf{K}}$ we understand a group of unimodular transformations $\mathbf{H}$ which satisfy the relation

$$\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X;\ \mathbf{F}(\sigma)\mathbf{H},\mathbf{T}(\tau)) = \underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X;\ \mathbf{F}(\sigma),\mathbf{T}(\tau)) \tag{2.5}$$

in the domain of the definition of $\mathfrak{h}_{\mathbf{K}}$. Because of Eq. (2.3) and by virtue of $\mathfrak{h}_{\hat{\mathbf{K}}}(X;\mathbf{F}(\sigma),\mathbf{T}(\tau)) = \mathfrak{h}_{\mathbf{K}}(X;\mathbf{F}(\sigma)\mathbf{G},\mathbf{T}(\tau))$ where $\hat{K} = \mathbf{G}K$, we obtain $\mathfrak{h}_{\mathbf{K}}(X;\mathbf{F}(\sigma)\mathbf{H},\mathbf{T}(\tau)) = \mathfrak{h}_{\hat{\mathbf{K}}}(X;\mathbf{F}(\sigma)\mathbf{H}\mathbf{G}^{-1},\mathbf{T}(\tau)) = \mathfrak{h}_{\hat{\mathbf{K}}}(X;\hat{\mathbf{F}}(\sigma)\mathbf{G}\mathbf{H}\mathbf{G}^{-1},\mathbf{T}(\tau)) = \mathfrak{h}_{\hat{\mathbf{K}}}(X;\hat{\mathbf{F}}(\sigma)\hat{\mathbf{H}},\mathbf{T}(\tau))$, where $\mathbf{F} = \hat{\mathbf{F}}\mathbf{G}$ and where we denote $\hat{\mathbf{H}} \equiv \mathbf{G}\mathbf{H}\mathbf{G}^{-1}$. It follows that the known relation $g_{\hat{\mathbf{K}}} = \mathbf{G}g_{\mathbf{K}}\mathbf{G}^{-1}$ between the isotropy groups $g_{\mathbf{K}}$ and $g_{\hat{\mathbf{K}}}$ holds also for the generalized simple materials.

Let us assume that there exists the local configuration $\mathbf{K}$ for which $g_{\mathbf{K}} = O$, where O is the full orthogonal group. Putting $\mathbf{H} = \mathbf{Q}^T$ in Eq. (2.5) and taking into account Eq. (1.3), we conclude that the relation

$$(\forall\mathbf{Q})\ [\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X;\ \mathbf{Q}\mathbf{F}(\sigma)\mathbf{Q}^T,\mathbf{Q}\mathbf{T}(\tau)\mathbf{Q}^T) = \mathbf{0}] \tag{2.6}$$

holds for all $\mathbf{F}(\sigma)$, $\mathbf{T}(\tau)$, $\sigma,\tau\in(-\infty,t]$, satisfying Eq. (1.2). If for a particle X, there exist the local configuration $\mathbf{K}$ for which the latter condition is valid, then the generalized simple material at the particle X is said to be isotropic. From Eq. (2.3), making use of $(\forall\mathbf{Q})\,[\mathfrak{h}_{\mathbf{K}}(X;\mathbf{F}(\sigma)\mathbf{Q}^T,\mathbf{T}(\tau)) = 0]$ and putting $\mathbf{Q}^T = \mathbf{R}^T(t)$, because of $\mathbf{V}(t) = \mathbf{R}(t)\mathbf{U}(t)\mathbf{R}^T(t)$, we obtain the reduced form of the constitutive equation for isotropic generalized simple materials:

$$\underset{\sigma,\tau\in(-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}(X;\ \mathbf{U}_{(t)}(\sigma)\mathbf{V}(t),\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)) = \mathbf{0}. \tag{2.7}$$

From Eq. (2.7) it follows that for an arbitrary isotropic generalized simple material which is not simple (in the sense of the known definition, cf. [1] and Eq. (3.1)) the rotations can not be eliminated from the constitutive equations [cf. also the theory of the isotropic rate-type materials [1] and Eqs. (3.5.1), (3.6.1), below]. If $g_{\mathbf{K}} = u$, where u is a group of unimodular transformations, the form of Eq. (2.7) depends on $\mathbf{K}$ only by means of the mass density $\varrho_{\mathbf{K}}$ [1]; the class of materials defined by this equation will be called the class of generalized simple fluids. Analogously, we can define the generalized simple solids.

Following the approach outlined above we are able to generalize definitions and theorems of the theory of simple materials to the forms which also hold for generalized simple materials.

3. Special types of materials

Defining the functional $\mathfrak{h}_{\mathbf{K}}$ in the special form

$$\mathfrak{h}_{\mathbf{K}} \equiv \mathbf{T}(t) - \underset{\sigma\in(-\infty,t]}{\mathscr{G}_{\mathbf{K}}}(X;\ \mathbf{F}(\sigma)) \tag{3.1}$$

we obtain from (1.2) the well known definition of the simple materials. The relation (1.2) with the denotation (3.1) is called the principle of determinism for stresses, cf. [1]. Analogously, we can postulate the principle of determinism for deformations, putting

$$\mathfrak{h}_{\mathbf{K}} \equiv \mathbf{F}(t) - \underset{\tau\in(-\infty,t]}{\mathscr{H}_{\mathbf{K}}}(X;\mathbf{T}(\tau)). \tag{3.2}$$

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From (3.1) and (2.1) we obtain the known reduced form of the constitutive equations of simple materials. By virtue of (3.2) and (2.1) we arrive at

$$\mathbf{U}(t) = \underset{\tau \in (-\infty, t]}{\mathscr{H}_{\mathbf{K}}}\big(X; \mathbf{R}^T(\tau)\mathbf{T}(\tau)\mathbf{R}(\tau)\big). \tag{3.3}$$

Thus for materials defined by Eq. (3.2), the actual value of the pure deformation is determined by the history of stresses and that of rotations.

Now let us assume that the values of the functional $\mathfrak{h}_{\mathbf{K}}(X; \mathbf{F}(\sigma), \mathbf{T}(\tau))$, $\sigma, \tau \in (-\infty, t]$, depend only on the values of $\mathbf{F}(\sigma)$, $\mathbf{T}(\tau)$ for σ, τ very near to t. Let us also assume that the functions $\mathbf{F}(\sigma)$, $\mathbf{T}(\tau)$ have sufficiently many continuous time derivatives. Then the arguments $\mathbf{U}_{(t)}(\sigma)$, $\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)$ in Eq. (2.2) can be approximated by their Taylor expansions:

$$\mathbf{U}_{(t)}(\sigma) = \mathbf{1} + \frac{d}{d\sigma}\mathbf{U}_{(t)}(\sigma)|_{\sigma=t}(\sigma - t) + \ldots,$$

$$\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau) = \mathbf{T}(t) + \frac{d}{d\tau}[\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)]|_{\tau=t}(\tau - t) + \ldots$$

up to some orders r, s, respectively. Substituting the right hand sides of the foregoing relations into Eq. (2.2) we obtain the constitutive equation

$$\mathbf{f}_{\mathbf{K}}\big(X; \mathbf{U}(t), \mathbf{R}^T(t)\mathbf{D}_n(t)\mathbf{R}(t), \mathbf{R}^T(t)\mathbf{T}(t)\mathbf{R}(t), \mathbf{R}^T(t)\mathring{\mathbf{T}}_m(t)\mathbf{R}(t)\big) = \mathbf{0}, \tag{3.4}$$

where $n = 1, \ldots, r$, $m = 1, \ldots, s$, $\mathbf{f}_{\mathbf{K}}$ is a known vector function and where

$$\mathbf{D}_n(t) \equiv \frac{d^n}{do^n}\mathbf{U}_{(t)}(\sigma)|_{\sigma=t}, \quad \mathring{\mathbf{T}}_m(t) \equiv \frac{d^m}{d\tau^m}[\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)]|_{\tau=t},$$

are the stretchings and the co-rotational stress rates, respectively. We have specified here the known rate-type materials [if the number of components of the vector $\mathbf{f}_{\mathbf{K}}$ is equal to six and if $\mathring{\mathbf{T}}_s$ can be calculated form (3.4)], which constitute the special case of the generalized simple materials. If $s \geqslant 1$, then, the constitutive equations of the form (3.4) cannot be obtained from the theory of simple materials [1]. Thus the rate-type materials for $s \geqslant 1$ are generalized simple materials but they are not simple materials.

If the values of the constitutive functional $\mathfrak{h}_{\mathbf{K}}$ depend only on the values of $\mathbf{T}(\tau)$ which are very near to t, and if the suitable regularity conditions hold, then the argument $\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)$ in Eq. (2.2) can be approximated by its Taylor expansion $\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau) = \mathbf{T}(t) + \mathring{\mathbf{T}}_1(t)(\tau - t) + \ldots$ up to some order s. Thus Eq. (2.2) can be replaced by the following one

$$\underset{\sigma \in (-\infty, t]}{\mathscr{F}_{\mathbf{K}}}\big(X; \mathbf{U}^*_{(t)}(\sigma)\mathbf{U}(t), \mathbf{R}^T(t)\mathbf{T}(t)\mathbf{R}(t), \mathbf{R}^T(t)\mathring{\mathbf{T}}_m(t)\mathbf{R}(t)\big) = \mathbf{0}, \tag{3.5}$$

where $m = 1, \ldots, s$ and $\mathscr{F}_{\mathbf{K}}$ is a function in arguments which depends on t only. The ideal materials defined by Eq. (3.5) will be called the stress rate-type materials. Using Eq. (2.5) for $\mathbf{H} = \mathbf{R}^T(t)$ and Eq. (2.6), by virtue of $\mathbf{R}(t)\mathbf{U}(t)\mathbf{R}^T(t) = \mathbf{V}(t)$, we can deduce from Eq. (3.5) that the relation

$$\underset{\sigma \in (-\infty, t]}{\mathscr{F}_{\mathbf{K}}}\big(X; \mathbf{U}_{(t)}(\sigma)\mathbf{V}(t), \mathbf{T}(t), \mathring{\mathbf{T}}_m(t)\big) = \mathbf{0}, \quad m = 1, \ldots, s, \tag{3.5.1}$$

holds for isotropic stress rate-type materials, provided that the local configuration $\mathbf{K}$ is properly chosen, cf. Sect. 2.

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Analogously, if the argument $\mathbf{U}_{(t)}(\sigma)$ in Eq. (2.2) can be approximated by the expansion $\mathbf{U}_{(t)}(\sigma) = \mathbf{1}+\mathbf{D}_1(t)(\sigma-t)+ \ldots$, up to some order r, then instead of (2.2) we shall obtain

$$(3.6) \qquad \underset{\tau\in(-\infty,t]}{\mathscr{G}_{\mathbf{K}}}\big(X;\ \mathbf{R}^T(t)\mathbf{D}_n(t)\mathbf{R}(t),\ \mathbf{U}(t),\ \mathbf{R}^T(t)\mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)\mathbf{R}(t)\big) = \mathbf{0},$$

where $n = 1, \ldots, r$ and where $\mathscr{G}_{\mathbf{K}}$ is a function in arguments which depend on t only. The ideal material defined by Eq. (3.6) will be called the strain rate-type material. By virtue of Eq. (2.6) and because of $\mathbf{U}(t) = \mathbf{R}^T(t)\mathbf{V}(t)\mathbf{R}(t)$, for the isotropic strain rate-type materials we obtain from (3.6)

$$(3.6.1) \qquad \underset{\tau\in(-\infty,t]}{\mathscr{G}_{\mathbf{K}}}\big(X;\mathbf{D}_n(t),\ \mathbf{V}(t),\ \mathbf{R}^T_{(t)}(\tau)\mathbf{T}(\tau)\mathbf{R}_{(t)}(\tau)\big) = \mathbf{0}, \quad n = 1, \ldots, r,$$

where $\mathscr{G}_{\mathbf{K}}$ is a function in arguments X, $\mathbf{D}_n(t)$, $\mathbf{V}(t)$.

We conclude that the known rate-type materials constitute a special case of the stress rate-type materials as well as the strain rate-type materials. In the general case all these materials are not simple. Following the approach outlined above we can also define the generalized simple materials of the integral type and we can formulate the principle of fading memory that holds for generalized simple materials.

We can also observe that the known theories of plastic materials can be obtained as special cases of the theory of generalized simple materials. One from Eqs. (1.2) will be specified as the algebraic equation for the stress components (it will represent the yield condition) and the remaining six will have the form (3.4) for $r = 1$ and $s = 0$.

The concept of the generalized simple material can be also applied in thermomechanics of the material continuum, by postulating the constitutive equations of the form

$$(3.7) \quad \underset{-\infty<\bar{s}\leqslant t}{\mathfrak{h}_{\mathbf{K}}}\big(X;\ \mathbf{F}(\sigma), \mathbf{T}(\tau), \theta(\tau_1), \mathbf{h}(\tau_2), \mathbf{g}(\tau_3), \varepsilon(\tau_4), \eta(\tau_5)\big) = \mathbf{0}; \quad \bar{s} \equiv \{\sigma, \tau, \tau_1, \ldots, \tau_5\},$$

where θ, $\mathbf{h}$, $\mathbf{g}$, ε, η, are the absolute temperature, the heat flux, the temperature gradient, the internal energy and the specific entropy, respectively. At the same time the principle of dissipation

$$(3.8) \qquad (\det\mathbf{F})^{-1}\varrho_R(\dot{\eta}\theta-\dot{\varepsilon})+\mathbf{T}\dot{\mathbf{F}}\mathbf{F}^{-1}+\frac{\mathbf{hg}}{\theta} \geqslant 0$$

has to be interpreted as the restriction imposed on the domain of the definition of the functional $\mathfrak{h}_{\mathbf{K}}$ in Eq. (3.7). This functional has to be also restricted by the principle of material frame indifference

(3.9)

$$(\forall\mathbf{Q}(s))\big[\underset{-\infty<\bar{s}\leqslant t}{\mathfrak{h}_{\mathbf{K}}}\big(X;\mathbf{Q}(\sigma)\mathbf{F}(\sigma),\mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^T(\tau),\theta(\tau_1),\mathbf{Q}(\tau_2)\mathbf{h}(\tau_2),\mathbf{Q}^T_{\mathbf{a}}(\tau_3)\mathbf{g}(\tau_3),\varepsilon(\tau_4),\eta(\tau_5)\big) = \mathbf{0},$$

where $\mathbf{Q}(s)$, $s \in (-\infty, t]$, is an orthogonal 3×3 matrix function.

The constitutive equations of the form (3.7), i.e., with the extra variables, can be also used in the pure mechanical problems. If the constitutive functional has the form

$$(3.10) \qquad \mathfrak{h}_{\mathbf{K}} \equiv \mathbf{T}(t)-\hat{\mathbf{T}}_{\mathbf{K}}(X;\mathbf{F}(t), \mathbf{A}(t)), \quad \dot{\mathbf{A}}(t)-\hat{\mathbf{A}}_{\mathbf{K}}(\mathbf{X};\mathbf{F}(t), \mathbf{A}(t)),$$

where $\hat{\mathbf{T}}_{\mathbf{K}}$, $\hat{\mathbf{A}}_{\mathbf{K}}$ are known functions, then we arrive at the concept of materials with internal variables [2].

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Now let us assume that on the motion of the body in the time interval (t_0, t_1) are imposed simple constraints $\gamma_\nu(\mathbf{X}; \mathbf{C}(\tau)) = 0$, $\mathbf{C} \equiv \mathbf{F}^T\mathbf{F}$, $\nu = 1, \ldots, m$, $m \leqslant \sigma$. Assuming that the forces maintaining the constraints do not work, we obtain the following constitutive equation of the generalized simple material with simple internal constraints

$$\underset{\sigma,\tau \in (-\infty,t]}{\mathfrak{h}_{\mathbf{K}}}\left(X;\ \mathbf{F}(\sigma),\ \mathbf{T}(\tau) - \sum_{\nu=1}^{m} \lambda^\nu(\tau)\mathbf{F}(\tau)\frac{\partial\gamma_\nu(\mathbf{X}; \mathbf{C}(\tau))}{\partial\mathbf{C}}\mathbf{F}^T(\tau)\right) = 0 \tag{3.11}$$

where $\lambda^\nu(\tau)$, $\nu = 1, \ldots, m$, are constraints functions (cf. [1], p. 71) such that $\lambda^\nu(\tau) \equiv 0$ for $\tau < t_0$ and for $\tau > t_1$. If the constraints are not simple, then we shall define the constrained generalized simple material by means of Eq. (1.2), provided that $\mathbf{T}(\tau)$ is an extra-stress tensor, [1], p. 71. In this case we have to apply the general approach to the problem of constrained material continua given in [3] and in the related papers.

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